

# Complementary Slackness

## Lemma 2

Assume a linear program  $P = \max\{c^T x \mid Ax \leq b; x \geq 0\}$  has solution  $x^*$  and its dual  $D = \min\{b^T y \mid A^T y \geq c; y \geq 0\}$  has solution  $y^*$ .

1. If  $x_j^* > 0$  then the  $j$ -th constraint in  $D$  is tight.
2. If the  $j$ -th constraint in  $D$  is not tight then  $x_j^* = 0$ .
3. If  $y_i^* > 0$  then the  $i$ -th constraint in  $P$  is tight.
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If we say that a variable  $x_j^*$  ( $y_i^*$ ) has slack if  $x_j^* > 0$  ( $y_i^* > 0$ ), (i.e., the corresponding variable restriction is not tight) and a constraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \leq y^{*T} A x^* \leq b^T y^*$$

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From the constraint of the dual it follows that  $y^T A \geq c^T$ . Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g.  $(y^T A - c^T)_j > 0$  (the  $j$ -th constraint in the dual is not tight) then  $x_j = 0$  (2.). The result for (1./3./4.) follows similarly.

# Interpretation of Dual Variables

- ▶ Brewer: find mix of ale and beer that maximizes profits

$$\begin{aligned} \max \quad & 13a + 23b \\ \text{s.t.} \quad & 5a + 15b \leq 480 \\ & 4a + 4b \leq 160 \\ & 35a + 20b \leq 1190 \\ & a, b \geq 0 \end{aligned}$$

- ▶ Entrepreneur: buy resources from brewer at minimum cost  
 $C, H, M$ : unit price for corn, hops and malt.

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Note that brewer won't sell (at least not all) if e.g.  
 $5C + 4H + 35M < 13$  as then brewing ale would be advantageous.

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# Interpretation of Dual Variables

## Marginal Price:

- ▶ How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- ▶ We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by  $\epsilon_C$ ,  $\epsilon_H$ , and  $\epsilon_M$ , respectively.

The profit increases to  $\max\{c^T x \mid Ax \leq b + \epsilon; x \geq 0\}$ . Because of strong duality this is equal to

$$\begin{array}{ll} \min & (b^T + \epsilon^T)y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

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If  $\epsilon$  is “small” enough then the optimum dual solution  $y^*$  might not change. Therefore the profit increases by  $\sum_i \epsilon_i y_i^*$ .

Therefore we can interpret the dual variables as **marginal prices**.

Note that with this interpretation, complementary slackness becomes obvious.

If the farmer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).

If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the farmer. Hence, it makes no sense to have a surplus of this resource. Therefore the slack must be zero.

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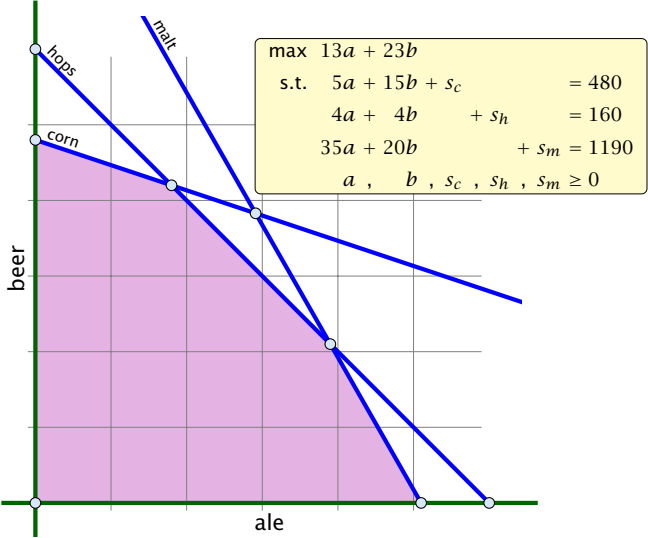
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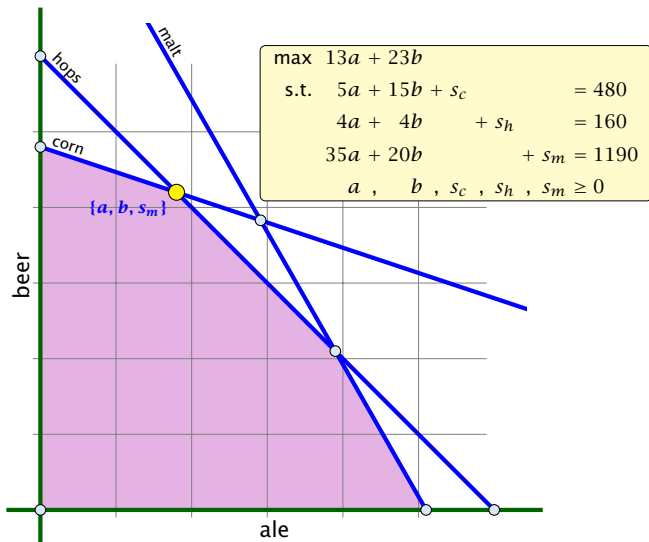
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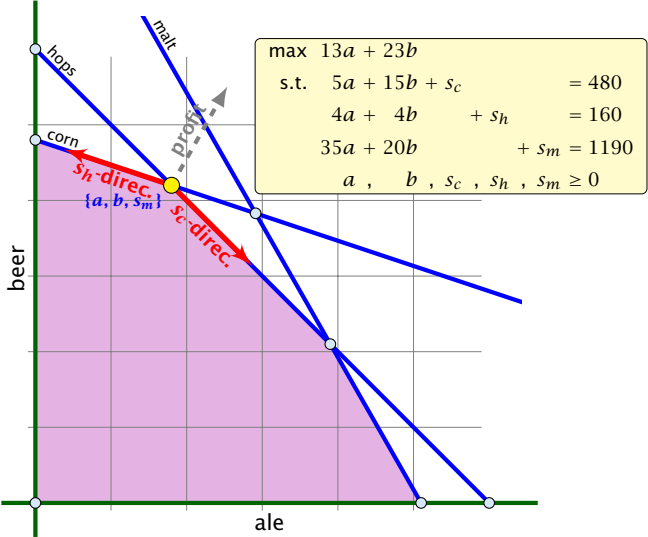
# Example



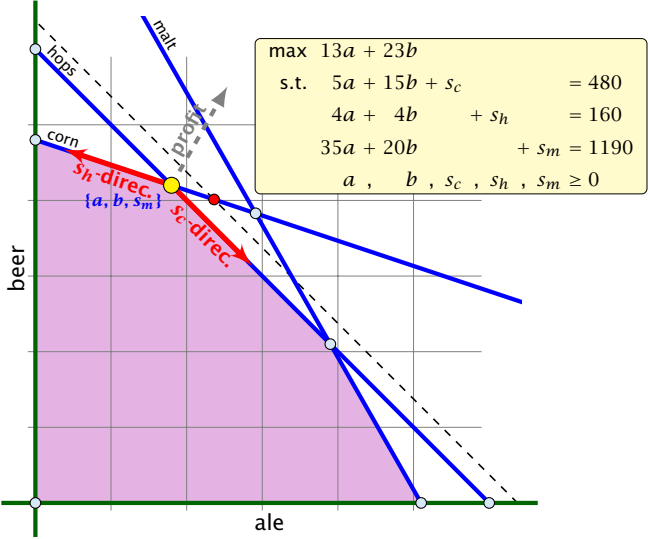
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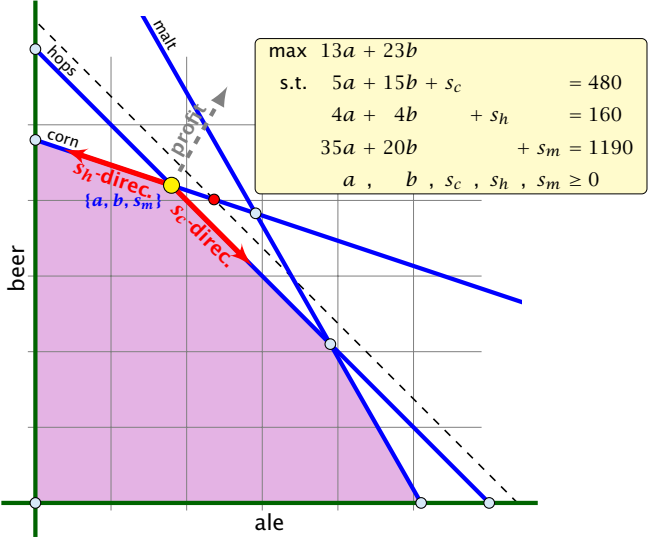
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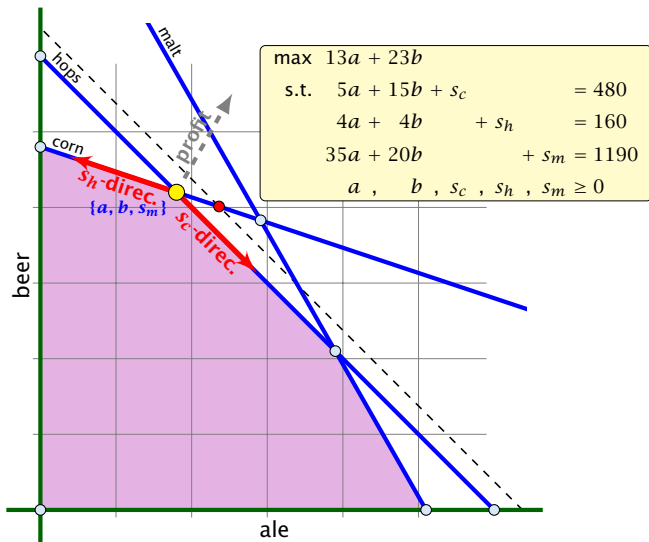
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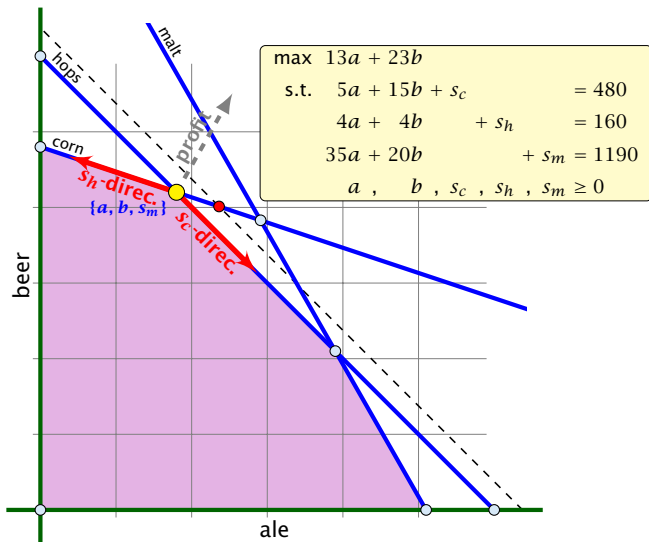
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The change in profit when increasing hops by one unit is

$$= c_B^T A_B^{-1} e_h.$$

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$$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h.$$



Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

# Flows

## Definition 3

An  $(s, t)$ -flow in a (complete) directed graph  $G = (V, V \times V, c)$  is a function  $f : V \times V \rightarrow \mathbb{R}_0^+$  that satisfies

1. For each edge  $(x, y)$

$$0 \leq f_{xy} \leq c_{xy} .$$

(capacity constraints)

2. For each  $v \in V \setminus \{s, t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} .$$

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## Definition 4

The **value of an  $(s, t)$ -flow  $f$**  is defined as

$$\text{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} .$$

Maximum Flow Problem:

Find an  $(s, t)$ -flow with maximum value.

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# LP-Formulation of Maxflow

$$\begin{array}{ll} \max & \sum_z f_{sz} - \sum_z f_{zs} \\ \text{s.t.} & \forall (z, w) \in V \times V \quad f_{zw} \leq c_{zw} \quad \ell_{zw} \\ & \forall w \neq s, t \quad \sum_z f_{zw} - \sum_z f_{wz} = 0 \quad p_w \\ & f_{zw} \geq 0 \end{array}$$

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} \ (x, y \neq s, t): \quad 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} \ (y \neq s, t): \quad 1\ell_{sy} \quad + 1p_y \geq 1 \\ & f_{xs} \ (x \neq s, t): \quad 1\ell_{xs} - 1p_x \quad \geq -1 \\ & f_{ty} \ (y \neq s, t): \quad 1\ell_{ty} \quad + 1p_y \geq 0 \\ & f_{xt} \ (x \neq s, t): \quad 1\ell_{xt} - 1p_x \quad \geq 0 \\ & f_{st}: \quad 1\ell_{st} \quad \geq 1 \\ & f_{ts}: \quad 1\ell_{ts} \quad \geq -1 \\ & \ell_{xy} \quad \geq 0 \end{array}$$

# LP-Formulation of Maxflow

$$\begin{array}{ll} \min & \sum_{(x,y)} c_{xy} l_{xy} \\ \text{s.t.} & f_{xy} (x, y \neq s, t) : 1l_{xy} - 1p_x + 1p_y \geq 0 \\ & f_{sy} (y \neq s, t) : 1l_{sy} - 1 + 1p_y \geq 0 \\ & f_{xs} (x \neq s, t) : 1l_{xs} - 1p_x + 1 \geq 0 \\ & f_{ty} (y \neq s, t) : 1l_{ty} - 0 + 1p_y \geq 0 \\ & f_{xt} (x \neq s, t) : 1l_{xt} - 1p_x + 0 \geq 0 \\ & f_{st} : 1l_{st} - 1 + 0 \geq 0 \\ & f_{ts} : 1l_{ts} - 0 + 1 \geq 0 \\ & l_{xy} \geq 0 \end{array}$$

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with  $p_t = 0$  and  $p_s = 1$ .



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$$\begin{array}{ll} \min & \sum_{(xy)} c_{xy} \ell_{xy} \\ \text{s.t.} & f_{xy} : 1\ell_{xy} - 1p_x + 1p_y \geq 0 \\ & \ell_{xy} \geq 0 \\ & p_s = 1 \\ & p_t = 0 \end{array}$$

We can interpret the  $\ell_{xy}$  value as assigning a length to every edge.

The value  $p_x$  for a variable, then can be seen as the distance of  $x$  to  $t$  (where the distance from  $s$  to  $t$  is required to be 1 since  $p_s = 1$ ).

The constraint  $p_x \leq \ell_{xy} + p_y$  then simply follows from triangle inequality ( $d(x, t) \leq d(x, y) + d(y, t) \Rightarrow d(x, t) \leq \ell_{xy} + d(y, t)$ ).

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means  $p_x = 1$  or  $p_x = 0$  for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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