## Complementary Slackness

**Lemma 2**

*Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ has solution $x^*$ and its dual $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ has solution $y^*$.*

1. *If $x_j^* > 0$ then the $j$-th constraint in $D$ is tight.*
2. *If the $j$-th constraint in $D$ is not tight than $x_j^* = 0$.*
3. *If $y_i^* > 0$ then the $i$-th constraint in $P$ is tight.*
4. *If the $i$-th constraint in $P$ is not tight than $y_i^* = 0$.*

If we say that a variable $x_j^*$ ($y_i^*$) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.



## Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

Because of strong duality we then get

$$c^T x^* = y^{*T} A x^* = b^T y^*$$

This gives e.g.

$$\sum_j (y^T A - c^T)_j x_j^* = 0$$

From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the $j$-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.



## Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

$$\begin{array}{lrcrl} \text{max} & 13a & + & 23b & \\ \text{s.t.} & 5a & + & 15b & \le 480 \\ & 4a & + & 4b & \le 160 \\ & 35a & + & 20b & \le 1190 \\ & & & a,b & \ge 0 \end{array}$$

- Entrepeneur: buy resources from brewer at minimum cost $C$, $H$, $M$: unit price for corn, hops and malt.

$$\begin{array}{lrcrcrl} \text{min} & 480C & + & 160H & + & 1190M & \\ \text{s.t.} & 5C & + & 4H & + & 35M & \ge 13 \\ & 15C & + & 4H & + & 20M & \ge 23 \\ & & & & & C,H,M & \ge 0 \end{array}$$

Note that brewer won't sell (at least not all) if e.g. $5C + 4H + 35M < 13$ as then brewing ale would be advantageous.

## Interpretation of Dual Variables

**Marginal Price:**

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C$, $\varepsilon_H$, and $\varepsilon_M$, respectively.

The profit increases to $\max\{c^T x \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^T + \epsilon^T) y & & \\ \text{s.t.} & A^T y & \ge & c \\ & y & \ge & 0 \end{array}$$

## Interpretation of Dual Variables

If $\epsilon$ is "small" enough then the optimum dual solution $y^*$ might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

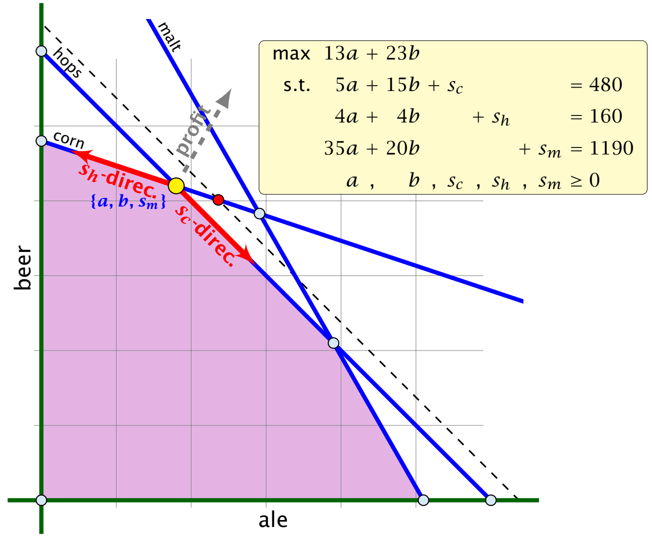
Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.



## Example




The change in profit when increasing hops by one unit is

$$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h .$$



Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.



## Flows

**Definition 3**

An $(s,t)$-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge $(x, y)$
$$0 \leq f_{xy} \leq c_{xy} .$$
(capacity constraints)
2. For each $v \in V \setminus \{s, t\}$
$$\sum_x f_{vx} = \sum_x f_{xv} .$$
(flow conservation constraints)

## Flows

**Definition 4**
The value of an $(s,t)$-flow $f$ is defined as

$$\operatorname{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

**Maximum Flow Problem:**
Find an $(s,t)$-flow with maximum value.



## LP-Formulation of Maxflow

$$\begin{array}{llrcll}
\text{max} & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \ge & 0 &
\end{array}$$

$$\begin{array}{lllcr}
\text{min} & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} \qquad\ + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} \qquad\ + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
 & f_{ts}: & 1\ell_{ts} & \ge & -1 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$



## LP-Formulation of Maxflow

$$\begin{array}{lllcr}
\text{min} & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - 1 + 1p_y & \ge & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + 1 & \ge & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - 0 + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \ge & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \ge & 0 \\
 & f_{ts}: & 1\ell_{ts} - 0 + 1 & \ge & 0 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$



## LP-Formulation of Maxflow

$$\begin{array}{lllcr}
\text{min} & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - p_s + 1p_y & \ge & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + p_s & \ge & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \ge & 0 \\
 & f_{st}: & 1\ell_{st} - p_s + p_t & \ge & 0 \\
 & f_{ts}: & 1\ell_{ts} - p_t + p_s & \ge & 0 \\
 & & \ell_{xy} & \ge & 0
\end{array}$$

with $p_t = 0$ and $p_s = 1$.

## LP-Formulation of Maxflow

$$\begin{array}{llrcl} \min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\ \text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\ & & \ell_{xy} & \geq & 0 \\ & & p_s & = & 1 \\ & & p_t & = & 0 \end{array}$$

We can interpret the $\ell_{xy}$ value as assigning a length to every edge.

The value $p_x$ for a variable, then can be seen as the distance of $x$ to $t$ (where the distance from $s$ to $t$ is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).



One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value $1$ on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.