Technique 4: The Greedy Algorithm

Algorithm 1 Greedy

$I \leftarrow \emptyset$
$\hat{S}_j \leftarrow S_j$ for all j
while I not a set cover **do**
  $\ell \leftarrow \arg\min_{j:\hat{S}_j \neq 0} \frac{w_j}{|\hat{S}_j|}$
  $I \leftarrow I \cup \{\ell\}$
  $\hat{S}_j \leftarrow \hat{S}_j - S_\ell$ for all j

In every round the Greedy algorithm takes the set that covers remaining elements in the most cost-effective way.

We choose a set such that the ratio between cost and still uncovered elements in the set is minimized.

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Lemma 4

Given positive numbers $a_1, \ldots, a_k$ *and* $b_1, \ldots, b_k$, *and* $S \subseteq \{1, \ldots, k\}$ *then*

$$\min_i \frac{a_i}{b_i} \leq \frac{\sum_{i \in S} a_i}{\sum_{i \in S} b_i} \leq \max_i \frac{a_i}{b_i}$$

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Let n_ℓ denote the number of elements that remain at the beginning of iteration ℓ. $n_1 = n = |U|$ and $n_{s+1} = 0$ if we need s iterations.

In the ℓ-th iteration

$$\min_j \frac{w_j}{|\hat{S}_j|} \le \frac{\sum_{j\in\mathrm{OPT}} w_j}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} = \frac{\mathrm{OPT}}{\sum_{j\in\mathrm{OPT}} |\hat{S}_j|} \le \frac{\mathrm{OPT}}{n_\ell}$$

since an optimal algorithm can cover the remaining n_ℓ elements with cost OPT.

Let $\hat{S}_j$ be a subset that minimizes this ratio. Hence, $w_j/|\hat{S}_j| \le \frac{\mathrm{OPT}}{n_\ell}$.

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Adding this set to our solution means $n_{\ell+1} = n_\ell - |\hat{S}_j|$.

$$w_j \le \frac{|\hat{S}_j|\mathrm{OPT}}{n_\ell} = \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT}$$

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$$
\begin{aligned}
\sum_{j\in I} w_j &\leq \sum_{\ell=1}^{s} \frac{n_\ell - n_{\ell+1}}{n_\ell} \cdot \mathrm{OPT} \\
&\leq \mathrm{OPT} \sum_{\ell=1}^{s} \left(\frac{1}{n_\ell} + \frac{1}{n_\ell - 1} + \cdots + \frac{1}{n_{\ell+1} + 1} \right) \\
&= \mathrm{OPT} \sum_{i=1}^{k} \frac{1}{i} \\
&= H_n \cdot \mathrm{OPT} \leq \mathrm{OPT}(\ln n + 1) \enspace .
\end{aligned}
$$

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A tight example: