Part II

Linear Programming

Brewery Problem

Brewery brews ale and beer.

- Production limited by supply of corn, hops and barley malt
- Recipes for ale and beer require different amounts of resources

	Corn (kg)	*Hops (kg)*	*Malt (kg)*	*Profit (€)*
ale (barrel)	5	4	35	13
beer (barrel)	15	4	20	23
supply	480	160	1190	

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How can brewer maximize profits?

- only brew ale: 34 barrels of ale $\Longrightarrow 442$ €
- only brew beer: 32 barrels of beer $\Longrightarrow 736$ €
- 7.5 barrels ale, 29.5 barrels beer $\Longrightarrow 776$ €
- 12 barrels ale, 28 barrels beer $\Longrightarrow 800$ €

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Brewery Problem

Linear Program

- Introduce [illegible] a and b that define how much ale and beer to produce.
- Choose the variables in such a way that the [illegible] (profit) is maximized.
- Make sure that no [illegible] (due to limited supply) are violated.

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\begin{array}{rrcrl}
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Standard Form LPs

LP in standard form:

- input: numbers a_{ij}, c_j, b_i
- output: numbers x_j
- n = #decision variables, m = #constraints
- maximize linear objective function subject to linear (in)equalities

$$\begin{array}{lrcll} \max & \sum_{j=1}^{n} c_j x_j & & & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & = & b_i & 1 \le i \le m \\ & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

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Standard Form

Add a slack variable to every constraint.

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It is easy to transform variants of LPs into (any) standard form:

- less or equal to equality:

- greater or equal to equality:

- min to max:

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- **less or equal to equality:**

$$a - 3b + 5c \leq 12 \implies \begin{aligned} a - 3b + 5c + s &= 12 \\ s &\geq 0 \end{aligned}$$

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Standard Form LPs

Observations:

- a linear program does not contain x^2, $\cos(x)$, etc.
- transformations between standard forms can be done efficiently and only change the size of the LP by a small constant factor
- for the standard minimization or maximization LPs we could include the nonnegativity constraints into the set of ordinary constraints; this is of course not possible for the standard form

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Fundamental Questions

Definition 1 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

Input size:

- n number of variables, m constraints, L number of bits to encode the input

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Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP?
- Is LP in P?

Input size:

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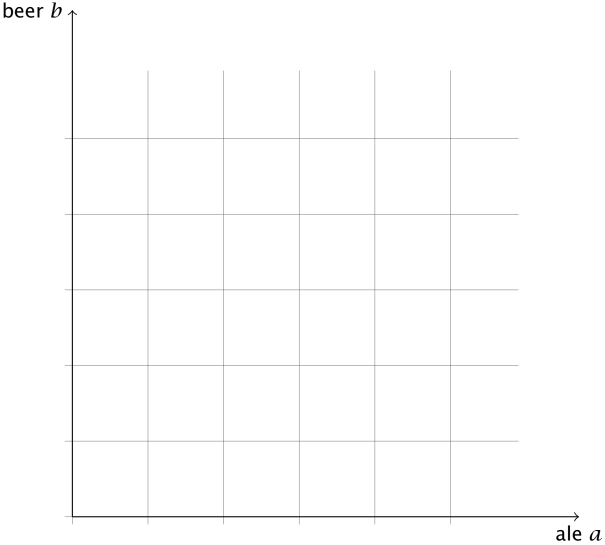

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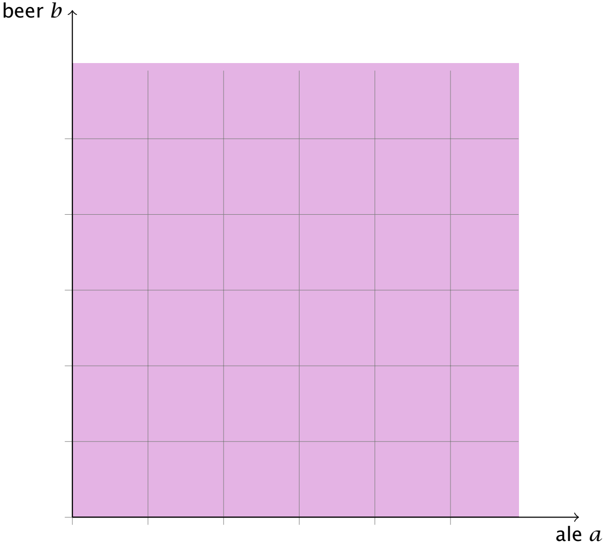

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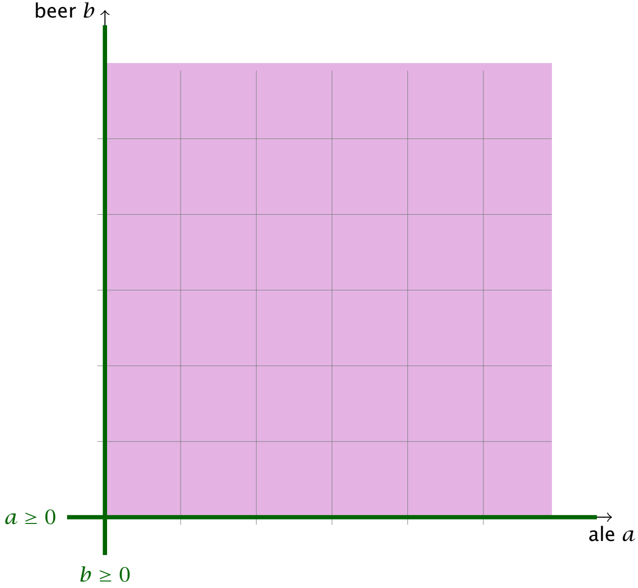

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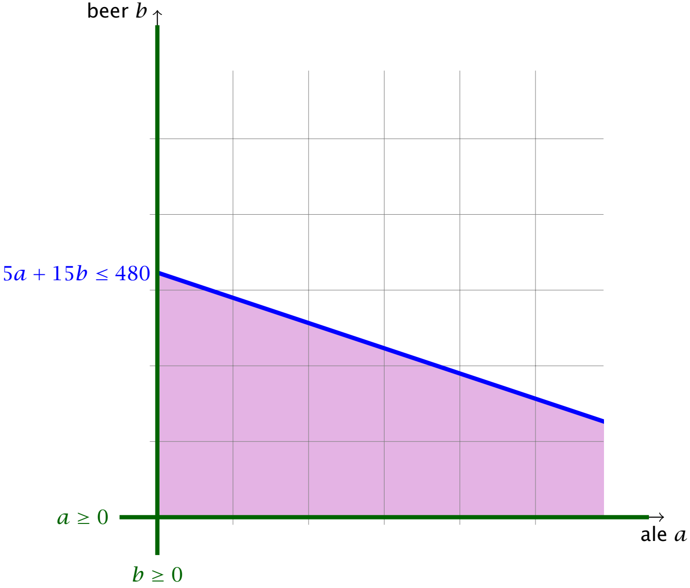

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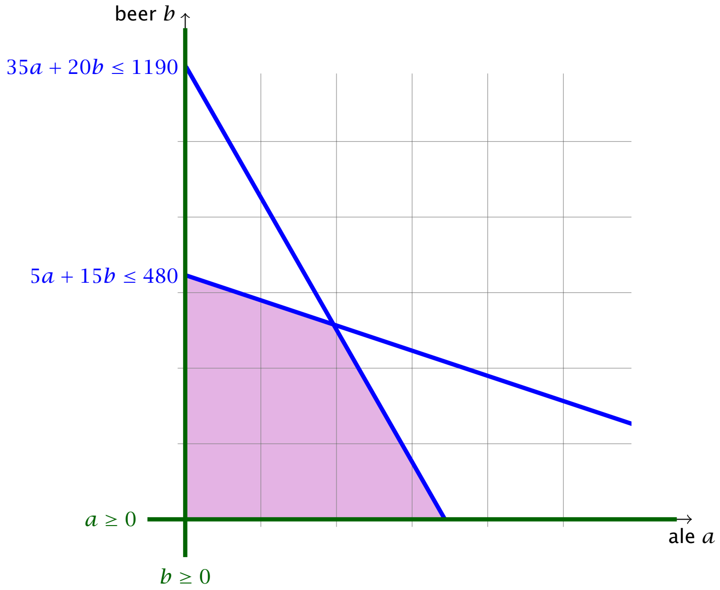

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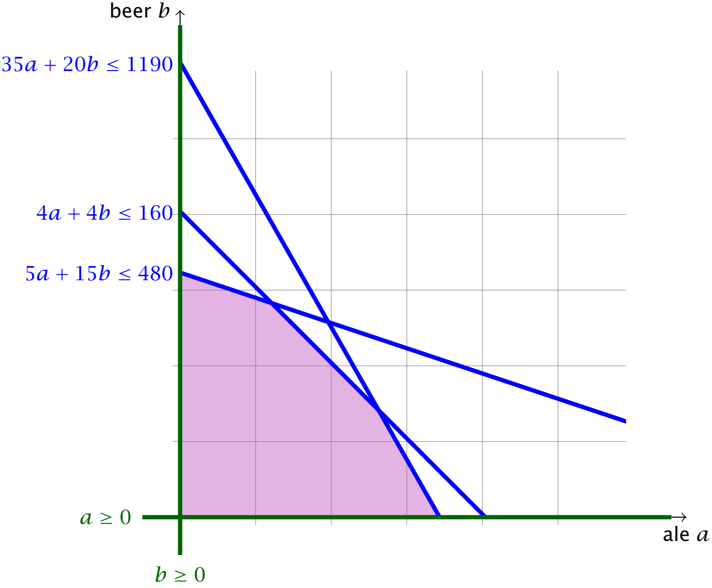

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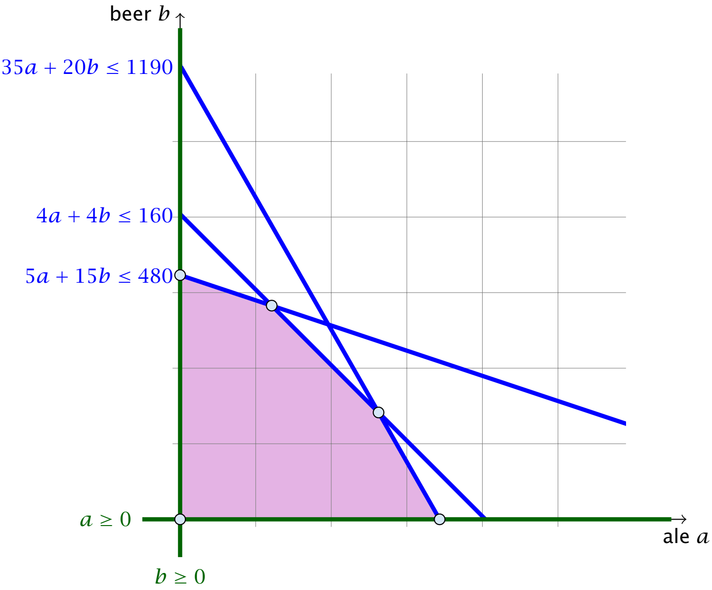

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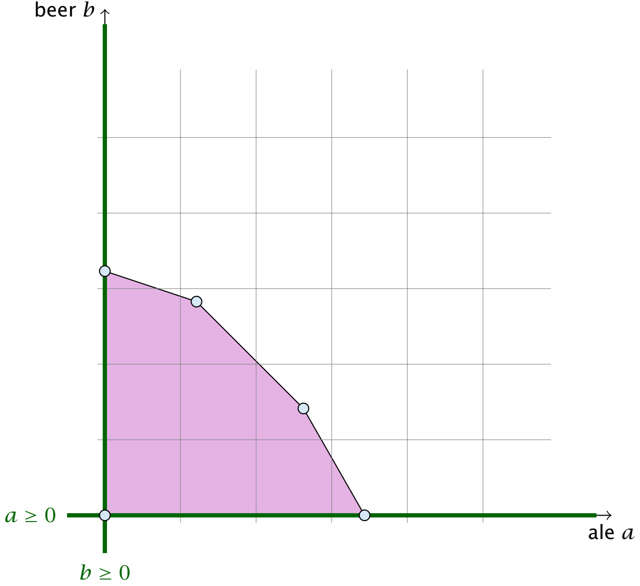

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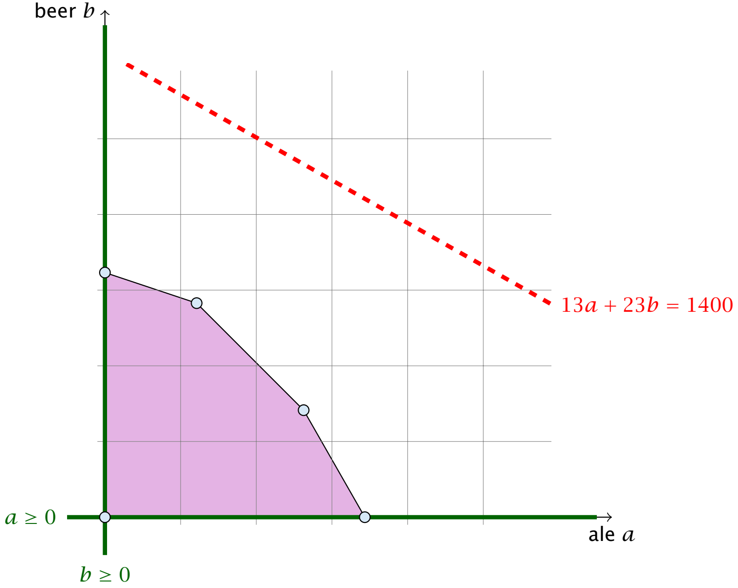

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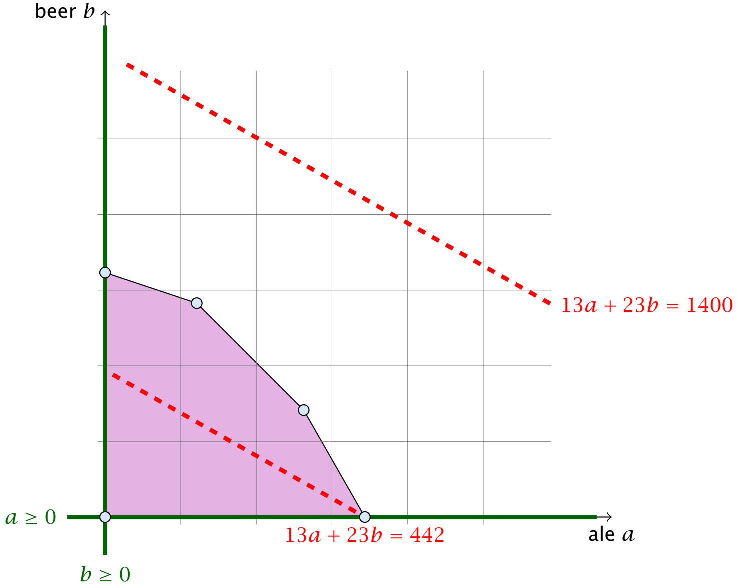

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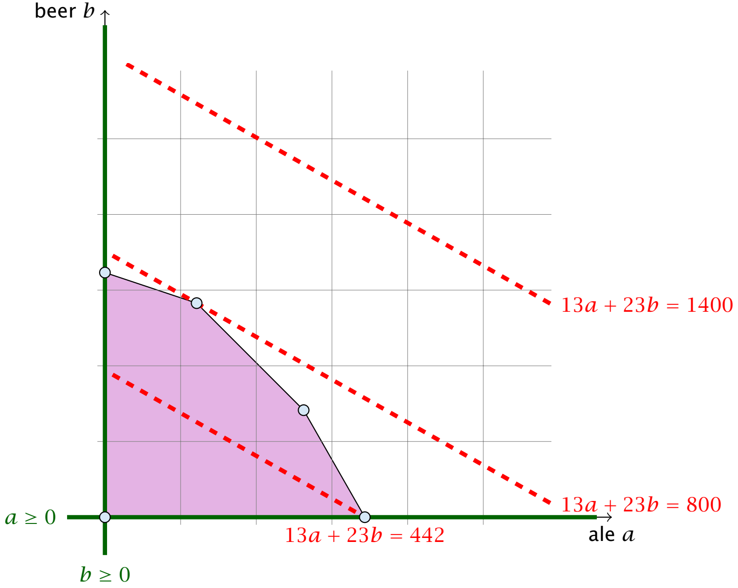

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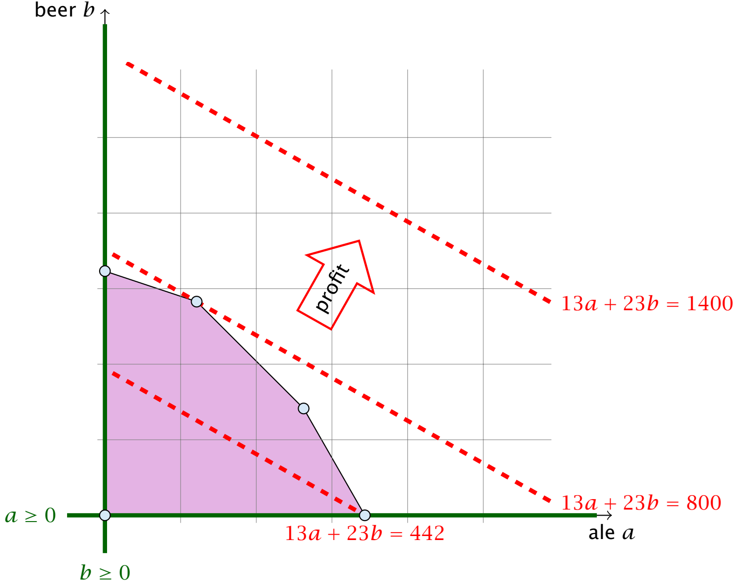

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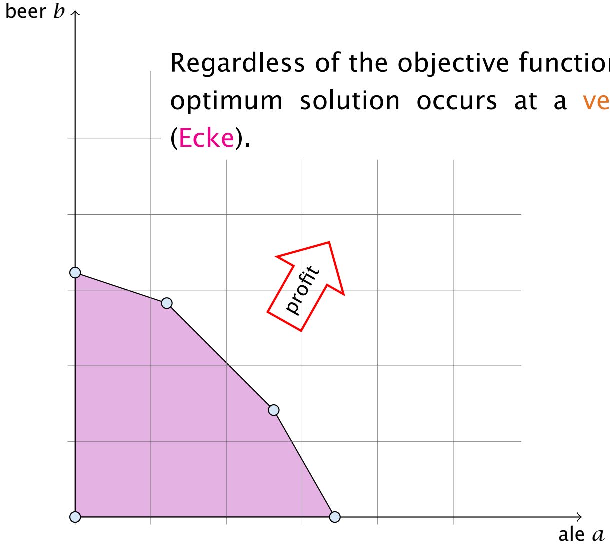

Regardless of the objective function an optimum solution occurs at a vertex (Ecke).

Definitions

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$P = \{x \mid Ax = b, x \geq 0\}$.

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Given points $x, y \in \mathbb{R}^n$, a point $z \in \mathbb{R}^n$ is a convex combination of x and y if

$$z = \lambda x + (1 - \lambda)y$$

for some $\lambda \in [0, 1]$.

Definition 3

A set $X \subseteq \mathbb{R}^n$ is convex if the convex combination of any two poins in X is also in X.

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A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \mathbb{R}^n$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Lemma 5

If $P \subseteq \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}$ convex than also

$$Q = \{x \in P \mid f(x) \leq t\}$$

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Definition 6

The dimension of a set $X \subseteq \mathbb{R}^n$ is the dimension of the vector space generated by vectors of the form $(x - y)$ with $x, y \in X$.

Definition 7

A set $H \subseteq \mathbb{R}^n$ is a hyperplane if $H = \{x \mid a^T x = b\}$, for $a \neq 0$.

Definition 8

A set $H' \subseteq \mathbb{R}^n$ is a (closed) halfspace if $H = \{x \mid a^T x \leq b\}$, for $a \neq 0$.

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Definitions

Definition 9

A polytop is a set $P \subseteq \mathbb{R}^n$ that is the convex hull of a finite set of points, i.e., $P = \mathrm{conv}(X)$ where

$$\mathrm{conv}(X) = \left\{ \sum_{i=1}^{\ell} \lambda_i x_i \mid \ell \in \mathbb{N}, x_1, \ldots, x_\ell \in X, \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}$$

and $|X| = c$.

Definitions

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A polyhedron is a set $P \subseteq \mathbb{R}^n$ that can be represented as the intersection of finitely many half-spaces $\{H(a_1, b_1), \dots, H(a_m, b_m)\}$, where

$$H(a_i, b_i) = \{x \in \mathbb{R}^n \mid a_i x \leq b_i\} \ .$$

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A polyhedron P is bounded if there exists B s.t. $\|x\|_2 \leq B$ for all $x \in P$.

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Definitions

Theorem 12
P is a bounded polyhedron iff P is a polytop.

Definition 13

Let $P \subseteq \mathbb{R}^n$, $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The hyperplane

$$H(a,b) = \{x \in \mathbb{R}^n \mid ax = b\}$$

is a supporting hyperplane of P if $\max\{ax \mid x \in P\} = b$.

Definition 14

Let $P \subseteq \mathbb{R}^n$. F is a face of P if $F = P$ or $F = P \cap H$ for some supporting hyperplane H.

Definition 15

Let $P \subseteq \mathbb{R}^n$.

- a face v is a vertex of P if $\{v\}$ is a face of P.
- a face e is an edge of P if e is a face and $\dim(e) = 1$.
- a face F is a facet of P if F is a face and $\dim(F) = \dim(P) - 1$.

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Equivalent definition for vertex:

Definition 16

Given polyhedron P. A point $x \in P$ is a vertex if $\exists c \in \mathbb{R}^n$ such that $c^T x < c^T y$, for all $y \in P$.

Definition 17

Given polyhedron P. A point $x \in P$ is an extreme point if $\nexists a, b \neq x$, $a, b \in P$, with $\lambda a + (1-\lambda) b = x$ for $\lambda \in [0,1]$.

Lemma 18

A vertex is also an extreme point.

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Observation

The feasible region of an LP is a Polyhedron.

Convex Sets

Theorem 19

If there exists an optimal solution to an LP (in standard form) then there exists an optimum solution that is an extreme point.

Proof

- suppose x is optimal solution that is not extreme point
- there exists direction $d \neq 0$ such that $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- Wlog. assume $c^T d \geq 0$ (by taking either d or $-d$)
- Consider $x + \lambda d$, $\lambda > 0$

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Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

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- $x + \lambda' d$ has one more zero component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

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- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
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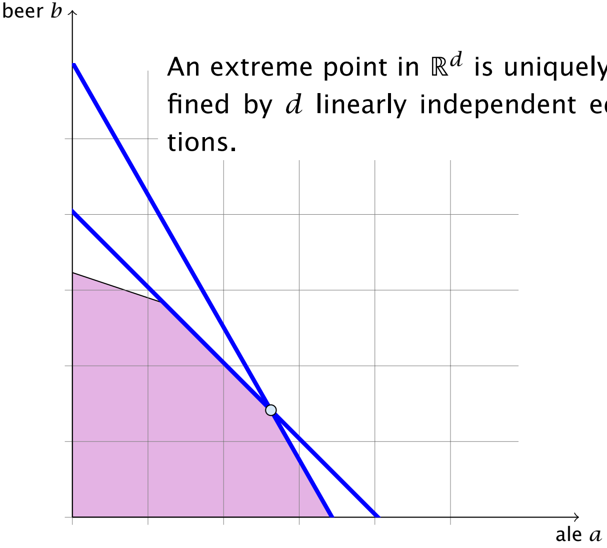
- $x + \lambda' d$ is feasible. Since $A(x + \lambda' d) = b$ and $x + \lambda' d \geq 0$
- $x + \lambda' d$ has one more zero-component ($d_k = 0$ for $x_k = 0$ as $x \pm d \in P$)
- $c^T x' = c^T(x + \lambda' d) = c^T x + \lambda' c^T d \geq c^T x$

Case 2. [$d_j \geq 0$ for all j and $c^T d > 0$]

- $x + \lambda d$ is feasible for all $\lambda \geq 0$ since $A(x + \lambda d) = b$ and $x + \lambda d \geq x \geq 0$
- as $\lambda \to \infty$, $c^T(x + \lambda d) \to \infty$ as $c^T d > 0$

Algebraic View



An extreme point in $\mathbb{R}^d$ is uniquely defined by d linearly independent equations.

Notation

Suppose $B \subseteq \{1 \dots n\}$ is a set of column-indices. Define A_B as the subset of columns of A indexed by B.

Theorem 20

*Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. Then x is extreme point **iff** A_B has linearly independent columns.*

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Proof ($\Leftarrow$)

- assume x is not extreme point
- there exists direction d s.t. $x \pm d \in P$
- $Ad = 0$ because $A(x \pm d) = b$
- define $B' = \{j \mid d_j \neq 0\}$
- $A_{B'}$ has linearly dependent columns as $Ad = 0$
- $d_j = 0$ for all j with $x_j = 0$ as $x \pm d \geq 0$
- Hence, $B' \subseteq B$, $A_{B'}$ is sub-matrix of A_B

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- there exists $d \neq 0$ such that $A_B d = 0$
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- now, $Ad = 0$ and $d_j = 0$ whenever $x_j = 0$
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Theorem 20

Let $P = \{x \mid Ax = b, x \geq 0\}$. For $x \in P$, define $B = \{j \mid x_j > 0\}$. If A_B has linearly independent columns then x is a vertex of P.

- define $c_j = \begin{cases} 0 & j \in B \\ 1 & j \notin B \end{cases}$
- then $c^T x = 0$ and $c^T y \geq 0$ for $y \in P$
- assume $c^T y = 0$; then $y_j = 0$ for all $j \notin B$
- $b = Ay = A_B y_B = Ax = A_B x_B$ gives that $A_B(x_B - Y_B) = 0$;
- this means that $x_B = y_B$ since A_B has linearly independent columns
- we get $y = x$
- hence, x is a vertex of P

Observation

For an LP we can assume wlog. that the matrix A has full row-rank. This means $\mathrm{rank}(A) = m$.

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C1 if now $b_1 = \sum_{i=2}^{m} \lambda_i \cdot b_i$ then

C2 if $b_1 \neq \sum_{i=2}^{m} \lambda_i \cdot b_i$ then the LP is infeasible, since for all x that fulfill constraints $A_2, \ldots, A_m$ we have

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From now on we will always assume that the constraint matrix of a standard form LP has full row rank.

Theorem 21

Given $P = \{x \mid Ax = b, x \geq 0\}$. x *is extreme point iff there exists* $B \subseteq \{1, \ldots, n\}$ *with* $|B| = m$ *and*

- A_B is non-singular
- $x_B = A_B^{-1} b \geq 0$
- $x_N = 0$

where $N = \{1, \ldots, n\} \setminus B$.

Proof

Take $B = \{j \mid x_j > 0\}$ and augment with linearly independent columns until $|B| = m$; always possible since $\operatorname{rank}(A) = m$.

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Basic Feasible Solutions

$x \in \mathbb{R}^n$ is called basic solution (Basislösung) if $Ax = b$ and $\text{rank}(A_J) = |J|$ where $J = \{j \mid x_j \neq 0\}$;

x is a basic **feasible** solution (gültige Basislösung) if in addition $x \geq 0$.

A basis (Basis) is an index set $B \subseteq \{1, \ldots, n\}$ with $\text{rank}(A_B) = m$ and $|B| = m$.

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Basic Feasible Solutions

A BFS fulfills the m equality constraints.

In addition, at least $n - m$ of the x_i's are zero. The corresponding non-negativity constraint is fulfilled with equality.

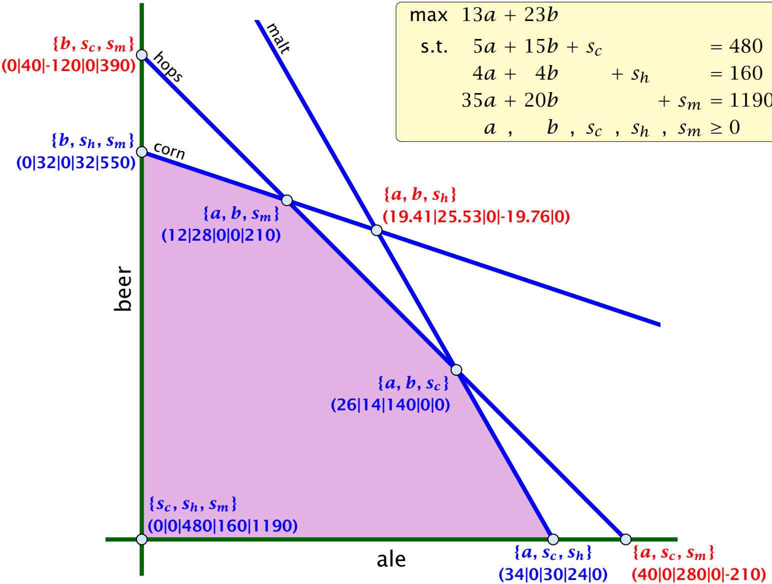
Fact:
In a BFS at least n constraints are fulfilled with equality.

Basic Feasible Solutions

Definition 22
For a general LP ($\min\{c^T x \mid Ax \geq b\}$) with n variables a point x is a basic feasible solution if x is feasible and there exist n (linearly independent) constraints that are tight.

Algebraic View

$$\begin{array}{llllllll} \max & 13a & + & 23b & & & & \\ \text{s.t.} & 5a & + & 15b & + s_c & & & = 480 \\ & 4a & + & 4b & & + s_h & & = 160 \\ & 35a & + & 20b & & & + s_m & = 1190 \\ & a & , & b & , s_c & , s_h & , s_m & \geq 0 \end{array}$$



Fundamental Questions

Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP? yes!
- Is LP in co-NP?
- Is LP in P?

Proof:

- Given a basis B we can compute the associated basis solution by calculating $A_B^{-1} b$ in polynomial time; then we can also compute the profit.

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Linear Programming Problem (LP)

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

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Observation

We can compute an optimal solution to a linear program in time $\mathcal{O}\left(\binom{n}{m}\cdot \mathrm{poly}(n,m)\right)$.

- there are only $\binom{n}{m}$ different bases.
- compute the profit of each of them and take the maximum

4 Simplex Algorithm

Enumerating all basic feasible solutions (BFS), in order to find the optimum is slow.

Simplex Algorithm [George Dantzig 1947]
Move from BFS to adjacent BFS, without decreasing objective function.

Two BFSs are called adjacent if the bases just differ in one variable.

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4 Simplex Algorithm

$$
\begin{array}{lrl}
\text{max} & 13a + 23b & \\
\text{s.t.} & 5a + 15b + s_c & = 480 \\
& 4a + 4b + s_h & = 160 \\
& 35a + 20b + s_m & = 1190 \\
& a, b, s_c, s_h, s_m & \geq 0
\end{array}
$$

$$
\begin{array}{lrl}
\text{max } Z & & \\
& 13a + 23b - Z & = 0 \\
& 5a + 15b + s_c & = 480 \\
& 4a + 4b + s_h & = 160 \\
& 35a + 20b + s_m & = 1190 \\
& a, b, s_c, s_h, s_m & \geq 0
\end{array}
$$

basis = $\{s_c, s_h, s_m\}$

$A = B = 0$

$Z = 0$

$s_c = 480$

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$s_m = 1190$

4 Simplex Algorithm

$$\begin{array}{lrcrcrcrcrcl} \text{max} & 13a & + & 23b & & & & & & & & \\ \text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\ & 4a & + & 4b & & & + & s_h & & & = & 160 \\ & 35a & + & 20b & & & & & + & s_m & = & 1190 \\ & a & , & b & , & s_c & , & s_h & , & s_m & \geq & 0 \end{array}$$

$$\begin{array}{rcrcrcrcrcrcl} \multicolumn{13}{l}{\text{max } Z} \\ 13a & + & 23b & & & & & & & - & Z & = & 0 \\ 5a & + & 15b & + & s_c & & & & & & & = & 480 \\ 4a & + & 4b & & & + & s_h & & & & & = & 160 \\ 35a & + & 20b & & & & & + & s_m & & & = & 1190 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array}$$

basis $= \{s_c, s_h, s_m\}$
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Pivoting Step

$$
\begin{array}{l}
\max\ Z \\
\begin{array}{rcrcccccl}
13a & + & 23b & & & & & -Z & = 0 \\
5a & + & 15b & + s_c & & & & & = 480 \\
4a & + & 4b & & + s_h & & & & = 160 \\
35a & + & 20b & & & + s_m & & & = 1190 \\
a & , & b & , s_c & , s_h & , s_m & & & \geq 0
\end{array}
\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
a = b = 0 \\
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Pivoting Step

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\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
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4a & + & 4b & & & + & s_h & & & & & = & 160 \\
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a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
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basis $= \{s_c, s_h, s_m\}$
$a = b = 0$
$Z = 0$

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- choose variable to bring into the basis
- chosen variable should have positive coefficient in objective function
- apply min-ratio test to find out by how much the variable can be increased
- pivot on row found by min-ratio test
- the existing basis variable in this row leaves the basis

Pivoting Step

$$
\begin{array}{l}
\max Z \\
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13a & + & 23b & & & & & & & - & Z & = & 0\\
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\begin{array}{l}
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13a & + & 23\boldsymbol{b} & & & & & & & - & Z & = & 0 \\
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\end{array}
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$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

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$s_m = 1190$

- Choose variable with coefficient ≥ 0 as entering variable.

$$\begin{array}{l} \max Z \\ \quad 13a + 23\boldsymbol{b} \qquad\qquad\qquad - Z = 0 \\ \quad 5a + 15\boldsymbol{b} + s_c \qquad\qquad\quad = 480 \\ \quad 4a + 4\boldsymbol{b} \quad + s_h \qquad\quad = 160 \\ \quad 35a + 20\boldsymbol{b} \qquad\quad + s_m \quad = 1190 \\ \quad a, \boldsymbol{b}, s_c, s_h, s_m \qquad\quad \geq 0 \end{array}$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

- Choose variable with coefficient ≥ 0 as entering variable.
- If we keep $a = 0$ and increase b from 0 to $\theta > 0$ s.t. all constraints ($Ax = b, x \geq 0$) are still fulfilled the objective value Z will strictly increase.

$$
\begin{array}{l}
\max Z \\
13a + 23\boldsymbol{b} \qquad\qquad\qquad\qquad - Z = 0 \\
5a + 15\boldsymbol{b} + s_c \qquad\qquad\qquad = 480 \\
4a + 4\boldsymbol{b} \qquad + s_h \qquad\qquad = 160 \\
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a\ ,\ \boldsymbol{b}\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad\qquad \geq 0
\end{array}
$$

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\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
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- For maintaining $Ax = b$ we need e.g. to set $s_c = 480 - 15\theta$.

$$\begin{array}{l}
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- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.

$$
\begin{array}{l}
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\quad 13a + 23b \qquad\qquad\qquad - Z = 0 \\
\quad 5a + 15\mathbf{b} + s_c \qquad\qquad\quad = 480 \\
\quad 4a + 4b \qquad + s_h \qquad\quad = 160 \\
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\end{array}
$$

$$
\begin{array}{l}
\text{basis} = \{s_c, s_h, s_m\} \\
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Z = 0 \\
s_c = 480 \\
s_h = 160 \\
s_m = 1190
\end{array}
$$

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- Choosing $\theta = \min\{480/15, 160/4, 1190/20\}$ ensures that in the new solution one current basic variable becomes 0, and no variable goes negative.
- The basic variable in the row that gives $\min\{480/15, 160/4, 1190/20\}$ becomes the leaving variable.

max Z

$$
\begin{array}{rcrcrcrcrcrcl}
13a & + & 23b & & & & & & & - & Z & = & 0 \\
5a & + & 15\mathbf{b} & + & s_c & & & & & & & = & 480 \\
4a & + & 4b & & & + & s_h & & & & & = & 160 \\
35a & + & 20b & & & & & + & s_m & & & = & 1190 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$

$a = b = 0$

$Z = 0$

$s_c = 480$

$s_h = 160$

$s_m = 1190$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

max Z

$$
\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcccccl}
13a & + & 23b & & & & & -Z & = 0 \\
5a & + & 15\boldsymbol{b} & + & s_c & & & & = 480 \\
4a & + & 4b & & & + s_h & & & = 160 \\
35a & + & 20b & & & & + s_m & & = 1190 \\
a & , & b & , & s_c & , s_h & , s_m & & \geq 0
\end{array}
\end{array}
$$

basis $= \{s_c, s_h, s_m\}$
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$Z = 0$

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Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$
\begin{array}{l}
\max Z \\
\begin{array}{rcrcrccccl}
\frac{16}{3}a & & & - & \frac{23}{15}s_c & & & -Z & = -736 \\
\frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & = 32 \\
\frac{8}{3}a & & & - & \frac{4}{15}s_c & + s_h & & & = 32 \\
\frac{85}{3}a & & & - & \frac{4}{3}s_c & & + s_m & & = 550 \\
a & , & b & , & s_c & , s_h & , s_m & & \geq 0
\end{array}
\end{array}
$$

basis $= \{b, s_h, s_m\}$
$a = s_c = 0$
$Z = 736$

$b = 32$
$s_h = 32$
$s_m = 550$

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcrcrcrcrcl} 13a & + & 23b & & & & & & & - & Z & = & 0 \\ 5a & + & 15\mathbf{b} & + & s_c & & & & & & & = & 480 \\ 4a & + & 4b & & & + & s_h & & & & & = & 160 \\ 35a & + & 20b & & & & & + & s_m & & & = & 1190 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

$$\begin{aligned} \text{basis} &= \{s_c, s_h, s_m\} \\ a &= b = 0 \\ Z &= 0 \\ s_c &= 480 \\ s_h &= 160 \\ s_m &= 1190 \end{aligned}$$

Substitute $b = \frac{1}{15}(480 - 5a - s_c)$.

$$\begin{array}{l} \text{max } Z \\ \begin{array}{rcrcrcrcrcrcl} \frac{16}{3}a & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\ \frac{1}{3}a & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\ \frac{8}{3}a & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\ \frac{85}{3}a & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\ a & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0 \end{array} \end{array}$$

$$\begin{aligned} \text{basis} &= \{b, s_h, s_m\} \\ a &= s_c = 0 \\ Z &= 736 \\ b &= 32 \\ s_h &= 32 \\ s_m &= 550 \end{aligned}$$

max Z

$$
\begin{array}{rcl}
\frac{16}{3}a \phantom{{}+b} - \frac{23}{15}s_c \phantom{{}+s_h+s_m} - Z &=& -736 \\
\frac{1}{3}a + b + \frac{1}{15}s_c &=& 32 \\
\frac{8}{3}a \phantom{{}+b} - \frac{4}{15}s_c + s_h &=& 32 \\
\frac{85}{3}a \phantom{{}+b} - \frac{4}{3}s_c \phantom{{}+s_h} + s_m &=& 550 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m &\geq& 0
\end{array}
$$

basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

$$\max Z$$

$$\begin{array}{rcrcrcrcrcrcl}
\frac{16}{3}\boldsymbol{a} & & & - & \frac{23}{15}s_c & & & & & - & Z & = & -736 \\
\frac{1}{3}\boldsymbol{a} & + & b & + & \frac{1}{15}s_c & & & & & & & = & 32 \\
\frac{8}{3}\boldsymbol{a} & & & - & \frac{4}{15}s_c & + & s_h & & & & & = & 32 \\
\frac{85}{3}\boldsymbol{a} & & & - & \frac{4}{3}s_c & & & + & s_m & & & = & 550 \\
\boldsymbol{a} & , & b & , & s_c & , & s_h & , & s_m & & & \geq & 0
\end{array}$$

basis = $\{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

$b = 32$

$s_h = 32$

$s_m = 550$

Choose variable a to bring into basis.

$$\begin{array}{l} \text{max } Z \\ \frac{16}{3}\boldsymbol{a} - \frac{23}{15}s_c - Z = -736 \\ \frac{1}{3}\boldsymbol{a} + b + \frac{1}{15}s_c = 32 \\ \frac{8}{3}\boldsymbol{a} - \frac{4}{15}s_c + s_h = 32 \\ \frac{85}{3}\boldsymbol{a} - \frac{4}{3}s_c + s_m = 550 \\ \boldsymbol{a}, b, s_c, s_h, s_m \geq 0 \end{array}$$

$$\begin{array}{l} \text{basis} = \{b, s_h, s_m\} \\ a = s_c = 0 \\ Z = 736 \\ b = 32 \\ s_h = 32 \\ s_m = 550 \end{array}$$

Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

$$
\begin{array}{l}
\text{max } Z \\
\begin{array}{rcccccl}
\frac{16}{3}a & & - \frac{23}{15}s_c & & & - Z & = -736 \\
\frac{1}{3}a + b & & + \frac{1}{15}s_c & & & & = 32 \\
\frac{8}{3}\boldsymbol{a} & & - \frac{4}{15}s_c & + s_h & & & = 32 \\
\frac{85}{3}a & & - \frac{4}{3}s_c & & + s_m & & = 550 \\
a, & b, & s_c, & s_h, & s_m & & \geq 0
\end{array}
\end{array}
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basis $= \{b, s_h, s_m\}$

$a = s_c = 0$

$Z = 736$

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Choose variable a to bring into basis.

Computing $\min\{3 \cdot 32, {}^{3\cdot 32}/_{8}, {}^{3\cdot 550}/_{85}\}$ means pivot on line 2.

Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\text{max } Z \\
\begin{aligned}
\tfrac{16}{3}a \qquad\quad - \tfrac{23}{15}s_c \qquad\qquad\qquad - Z &= -736 \\
\tfrac{1}{3}a + b + \tfrac{1}{15}s_c \qquad\qquad\qquad\qquad &= 32 \\
\tfrac{8}{3}\boldsymbol{a} \qquad\quad - \tfrac{4}{15}s_c + s_h \qquad\qquad\qquad &= 32 \\
\tfrac{85}{3}a \qquad\quad - \tfrac{4}{3}s_c \qquad\quad + s_m \qquad\quad &= 550 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad\qquad &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{b, s_h, s_m\} \\
a &= s_c = 0 \\
Z &= 736 \\
b &= 32 \\
s_h &= 32 \\
s_m &= 550
\end{aligned}
$$

Choose variable a to bring into basis.

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Substitute $a = \frac{3}{8}(32 + \frac{4}{15}s_c - s_h)$.

$$
\begin{array}{l}
\text{max } Z \\
\begin{aligned}
- s_c - 2s_h \qquad\qquad - Z &= -800 \\
b + \tfrac{1}{10}s_c - \tfrac{1}{8}s_h \qquad\qquad\qquad &= 28 \\
a \qquad - \tfrac{1}{10}s_c + \tfrac{3}{8}s_h \qquad\qquad\qquad &= 12 \\
\tfrac{3}{2}s_c - \tfrac{85}{8}s_h + s_m \qquad\qquad &= 210 \\
a\ ,\ b\ ,\ s_c\ ,\ s_h\ ,\ s_m \qquad\qquad &\geq 0
\end{aligned}
\end{array}
$$

$$
\begin{aligned}
\text{basis} &= \{a, b, s_m\} \\
s_c &= s_h = 0 \\
Z &= 800 \\
b &= 28 \\
a &= 12 \\
s_m &= 210
\end{aligned}
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4 Simplex Algorithm

Pivoting stops when all coefficients in the objective function are non-positive.

Solution is optimal:

- any feasible solution satisfies all equations in the tableaux
- in particular: $Z = 800 - s_c - 2s_h$, $s_c \geq 0$, $s_h \geq 0$
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Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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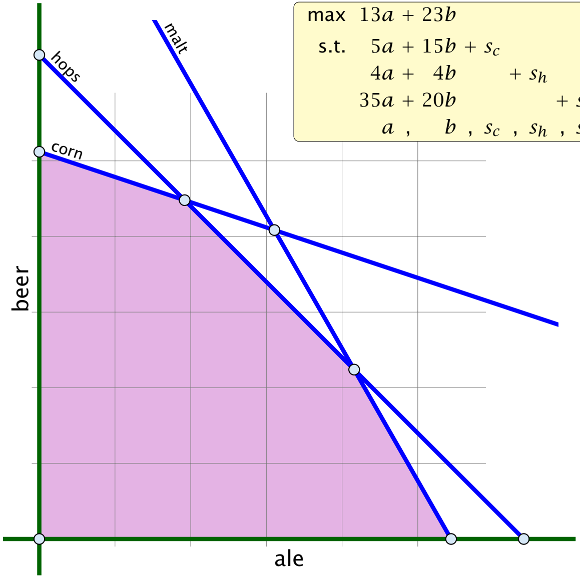
Geometric View of Pivoting



$$\begin{array}{lrl} \text{max} & 13a + 23b & \\ \text{s.t.} & 5a + 15b + s_c & = 480 \\ & 4a + 4b + s_h & = 160 \\ & 35a + 20b + s_m & = 1190 \\ & a, b, s_c, s_h, s_m & \geq 0 \end{array}$$

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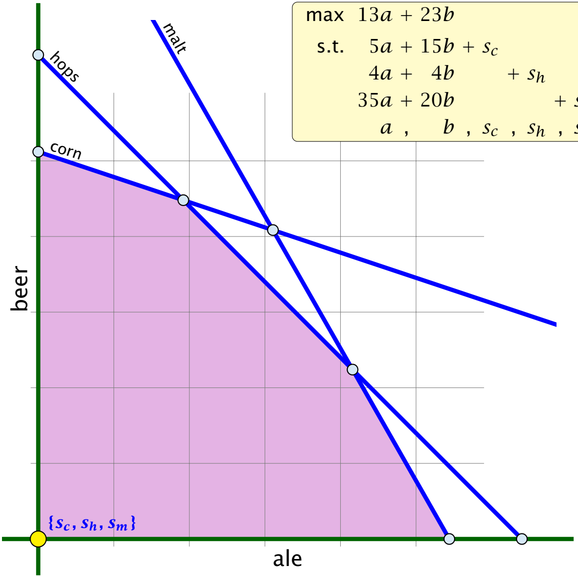

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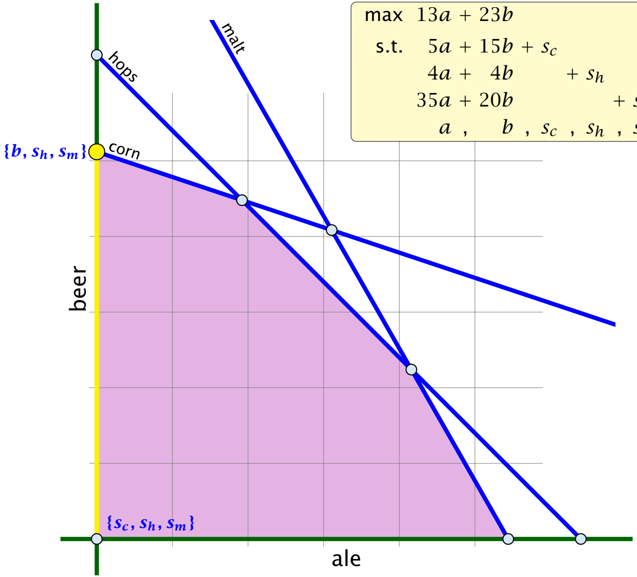

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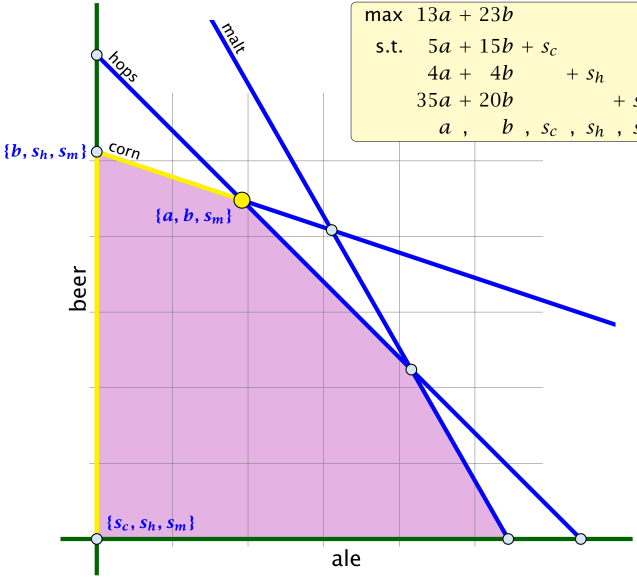

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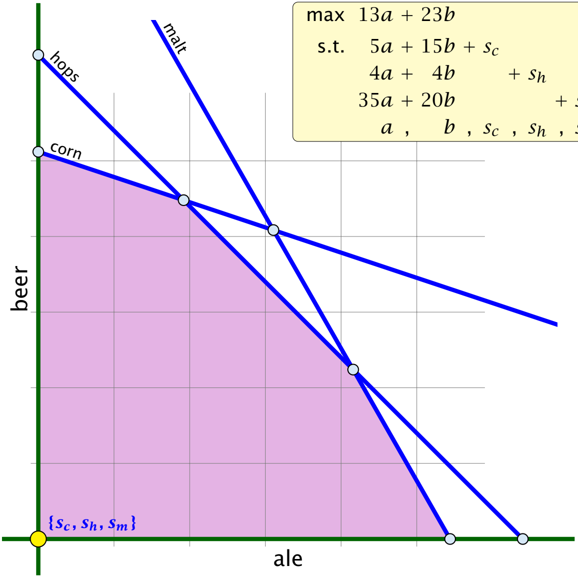

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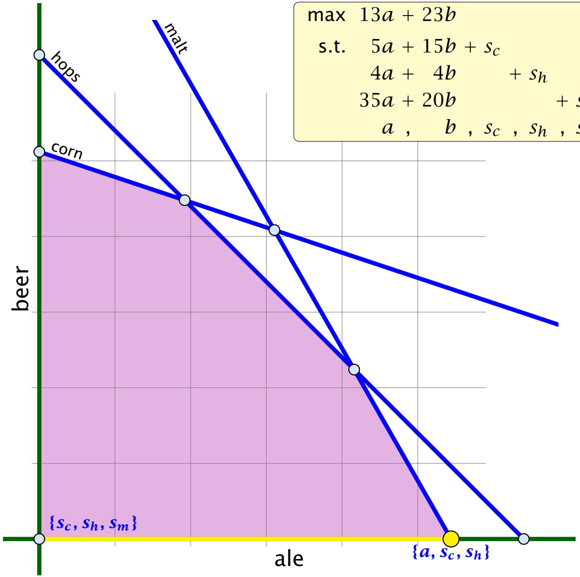

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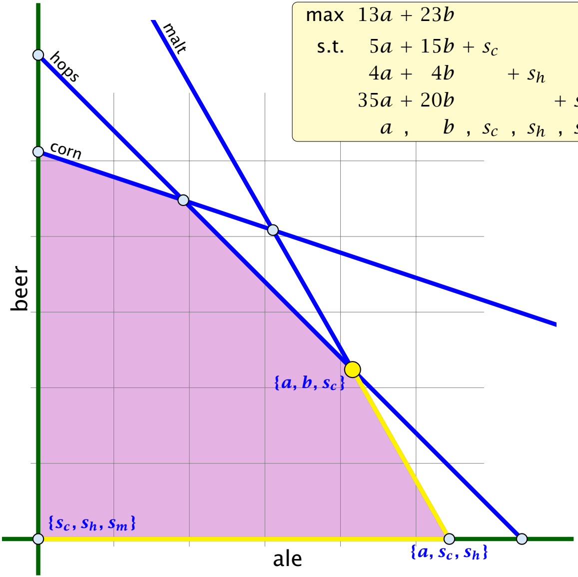

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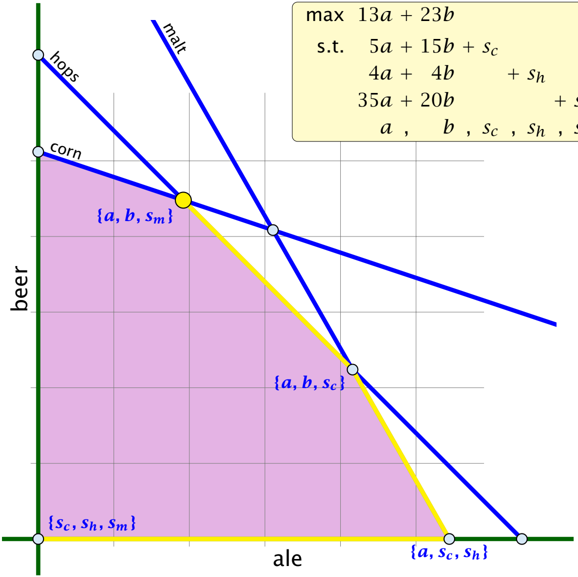

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Algebraic Definition of Pivoting

- Given basis B with BFS x^*.
- Choose index $j \notin B$ in order to increase x_j^* from 0 to $\theta > 0$.
  - Other non-basis variables should stay at 0.
  - Basis variables change to maintain feasibility.
- Go from x^* to $x^* + \theta \cdot d$.

Requirements for d:

- $d_j = 1$ (normalization)
- $d_\ell = 0$, $\ell \notin B$, $\ell \neq j$
- $A(x^* + \theta d) = b$ must hold. Hence $Ad = 0$.
- Altogether: $A_B d_B + A_{*j} = Ad = 0$, which gives $d_B = -A_B^{-1} A_{*j}$.

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Algebraic Definition of Pivoting

Definition 23 (j-th basis direction)

Let B be a basis, and let $j \notin B$. The vector d with $d_j = 1$ and $d_\ell = 0, \ell \notin B, \ell \neq j$ and $d_B = -A_B^{-1} A_{*j}$ is called the j-th basis direction for B.

Going from x^* to $x^* + \theta \cdot d$ the objective function changes by

$$\theta \cdot c^T d = \theta(c_j - c_B^T A_B^{-1} A_{*j})$$

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Definition 24 (Reduced Cost)
For a basis B the value

$$\tilde{c}_j = c_j - c_B^T A_B^{-1} A_{*j}$$

is called the reduced cost for variable x_j.

Note that this is defined for every j. If $j \in B$ then the above term is 0.

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Let our linear program be

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The simplex tableaux for basis B is

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4 Simplex Algorithm

Questions:

- What happens if the min ratio test fails to give us a value θ by which we can safely increase the entering variable?
- How do we find the initial basic feasible solution?
- Is there always a basis B such that

$$(c_N^T - c_B^T A_B^{-1} A_N) \le 0 \ ?$$

  Then we can terminate because we know that the solution is optimal.

- If yes how do we make sure that we reach such a basis?

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Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

What does it mean that the ratio b_i/A_{ie} (and hence A_{ie}) is negative for a constraint?

This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

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This means that the corresponding basic variable will increase if we increase b. Hence, there is no danger of this basic variable becoming negative

What happens if **all** b_i/A_{ie} are negative? Then we do not have a leaving variable. Then the LP is unbounded!

Min Ratio Test

The min ratio test computes a value $\theta \geq 0$ such that after setting the entering variable to θ the leaving variable becomes 0 and all other variables stay non-negative.

For this, one computes b_i/A_{ie} for all constraints i and calculates the minimum positive value.

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The objective function does not decrease during one iteration of the simplex-algorithm.

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The objective function may not increase!

Because a variable x_ℓ with $\ell \in B$ is already 0.

The set of inequalities is degenerate (also the basis is degenerate).

Definition 25 (Degeneracy)
A BFS x^* is called degenerate if the set $J = \{j \mid x_j^* > 0\}$ fulfills $|J| < m$.

It is possible that the algorithm cycles, i.e., it cycles through a sequence of different bases without ever terminating. Happens, very rarely in practise.

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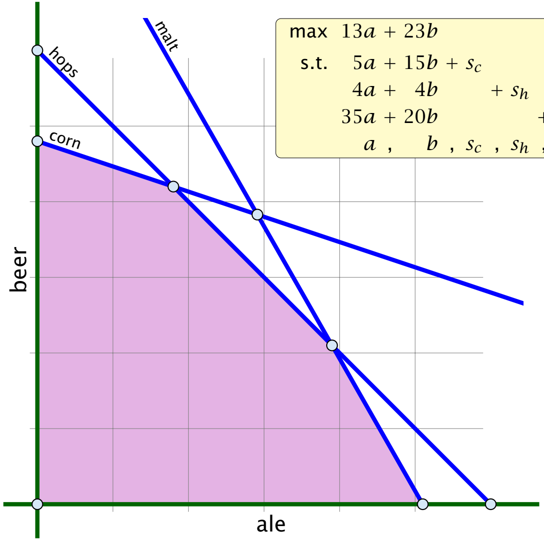
Non Degenerate Example



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\begin{array}{lrcrcccccl}
\text{max} & 13a & + & 23b & & & & & & \\
\text{s.t.} & 5a & + & 15b & + & s_c & & & = & 480 \\
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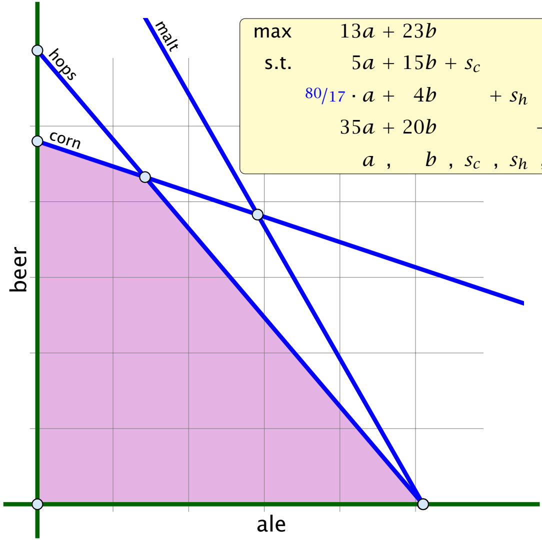

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\text{max} & 13a & + & 23b & & & & \\
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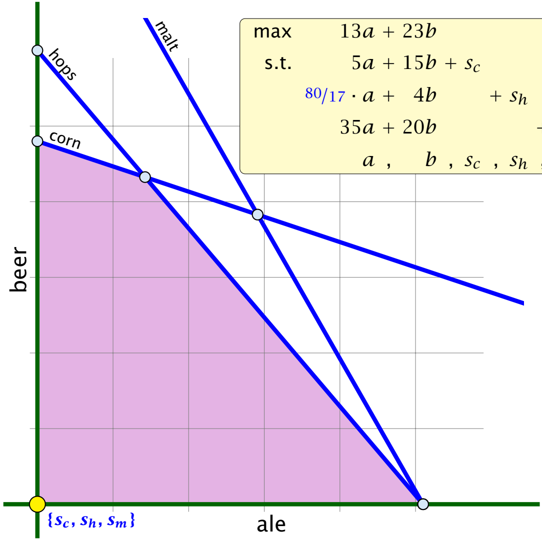

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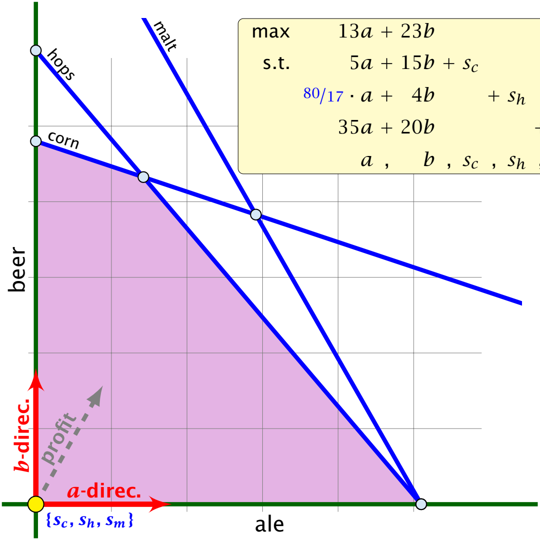

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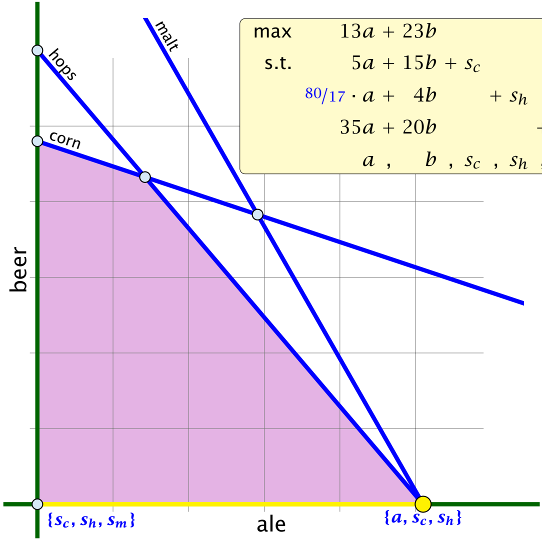

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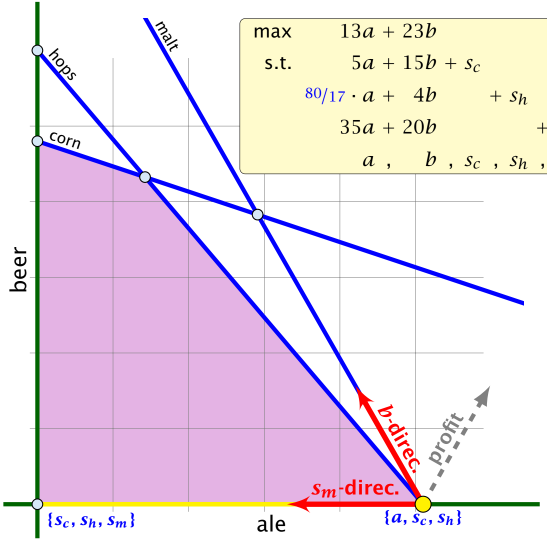

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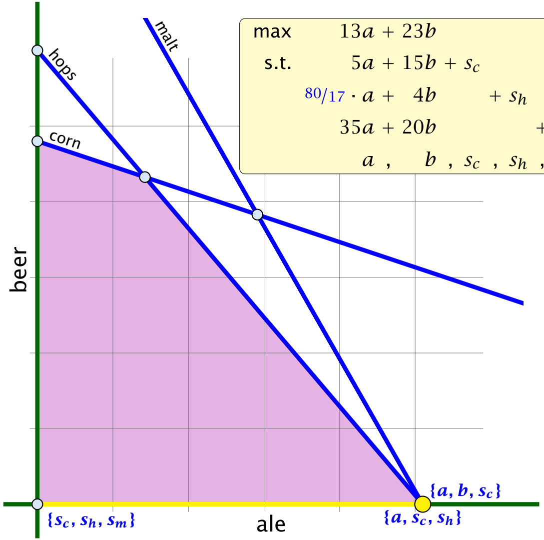

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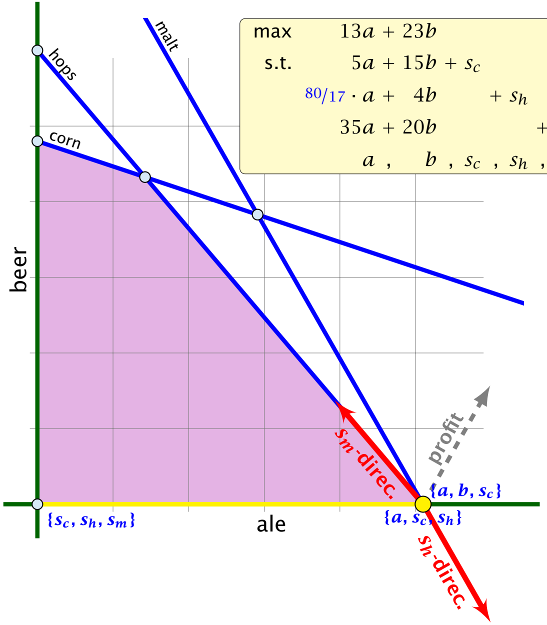

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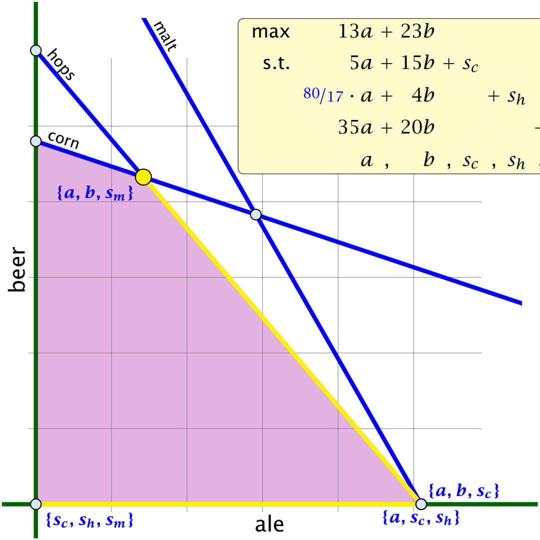

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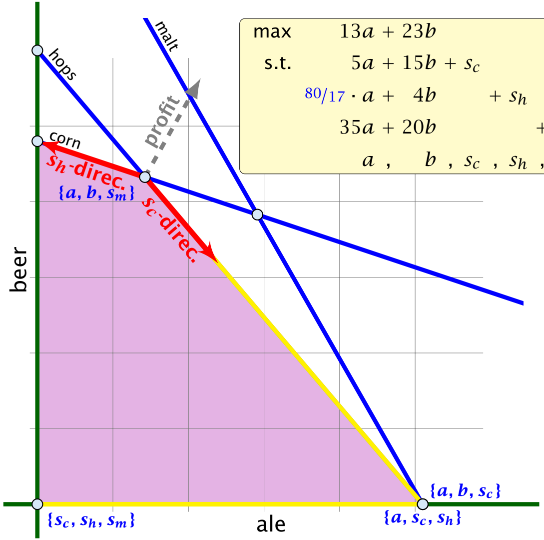

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Summary: How to choose pivot-elements

- We can choose a column e as an entering variable if $\tilde{c}_e > 0$ ($\tilde{c}_e$ is reduced cost for x_e).
- The standard choice is the column that maximizes $\tilde{c}_e$.
- If $A_{ie} \leq 0$ for all $i \in \{1, \ldots, m\}$ then the maximum is not bounded.
- Otw. choose a leaving variable ℓ such that $b_\ell / A_{\ell e}$ is minimal among all variables i with $A_{ie} > 0$.
- If several variables have minimum $b_\ell / A_{\ell e}$ you reach a degenerate basis.
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Termination

What do we have so far?

Suppose we are given an initial feasible solution to an LP. If the LP is non-degenerate then Simplex will terminate.

Note that we either terminate because the min-ratio test fails and we can conclude that the LP is unbounded, or we terminate because the vector of reduced cost is non-positive. In the latter case we have an optimum solution.

How do we come up with an initial solution?

- $Ax \leq b, x \geq 0$, and $\boldsymbol{b} \geq \boldsymbol{0}$.
- The standard slack from for this problem is $Ax + Is = b, x \geq 0, s \geq 0$, where s denotes the vector of slack variables.
- Then $s = b$, $x = 0$ is a basic feasible solution (how?).
- We directly can start the simplex algorithm.

How do we find an initial basic feasible solution for an arbitrary problem?

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Two phase algorithm

Suppose we want to maximize $c^T x$ s.t. $Ax = b, x \geq 0$.

1. Multiply all rows with $b_i < 0$ by -1.
2. maximize $-\sum_i v_i$ s.t. $Ax + Iv = b$, $x \geq 0$, $v \geq 0$ using Simplex. $x = 0$, $v = b$ is initial feasible.
3. If $\sum_i v_i > 0$ then the original problem is infeasible.
4. Otw. you have $x \geq 0$ with $Ax = b$.
5. From this you can get basic feasible solution.
6. Now you can start the Simplex for the original problem.

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Optimality

Lemma 26

Let B be a basis and x^* a BFS corresponding to basis B. $\tilde{c} \leq 0$ implies that x^* is an optimum solution to the LP.

Duality

How do we get an upper bound to a maximization LP?

$$
\begin{array}{rrcrl}
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\text{s.t.} & 5a & + & 15b & \le 480 \\
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 & 35a & + & 20b & \le 1190 \\
 & & & a, b & \ge 0
\end{array}
$$

Note that a lower bound is easy to derive. Every choice of $a, b \ge 0$ gives us a lower bound (e.g. $a = 12, b = 28$ gives us a lower bound of 800).

If you take a conic combination of the rows (multiply the i-th row with $y_i \ge 0$) such that $\sum_i y_i a_{ij} \ge c_j$ then $\sum_i y_i b_i$ will be an upper bound.

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Definition 27
Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ be a linear program P (called the primal linear program).

The linear program D defined by

$$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$$

is called the dual problem.

Duality

Lemma 28

The dual of the dual problem is the primal problem.

Proof:

- $w = \min\{b^T y \mid A^T y \geq c, y \geq 0\}$
- $w = -\max\{-b^T y \mid -A^T y \leq -c, y \geq 0\}$

The dual problem is

- $z = -\min\{-c^T x \mid -Ax \geq -b, x \geq 0\}$
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Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

Theorem 29 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

Weak Duality

Let $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$ and
$w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$ be a primal dual pair.

x is primal feasible iff $x \in \{x \mid Ax \le b, x \ge 0\}$

y is dual feasible, iff $y \in \{y \mid A^T y \ge c, y \ge 0\}$.

Theorem 29 (Weak Duality)

Let $\hat{x}$ be primal feasible and let $\hat{y}$ be dual feasible. Then

$$c^T \hat{x} \le z \le w \le b^T \hat{y} \ .$$

Weak Duality

$A^T\hat{y} \geq c \Rightarrow \hat{x}^TA^T\hat{y} \geq \hat{x}^Tc \ (\hat{x} \geq 0)$

$A\hat{x} \leq b \Rightarrow y^TA\hat{x} \leq \hat{y}^Tb \ (\hat{y} \geq 0)$

This gives

$$c^T\hat{x} \leq \hat{y}^TA\hat{x} \leq b^T\hat{y} \ .$$

Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^Ty = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

Weak Duality

$$A^T\hat{y} \geq c \Rightarrow \hat{x}^T A^T \hat{y} \geq \hat{x}^T c \; (\hat{x} \geq 0)$$

$$A\hat{x} \leq b \Rightarrow y^T A\hat{x} \leq \hat{y}^T b \; (\hat{y} \geq 0)$$

This gives

$$c^T\hat{x} \leq \hat{y}^T A\hat{x} \leq b^T\hat{y} \; .$$

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Since, there exists primal feasible $\hat{x}$ with $c^T\hat{x} = z$, and dual feasible $\hat{y}$ with $b^T y = w$ we get $z \leq w$.

If P is unbounded then D is infeasible.

The following linear programs form a primal dual pair:

$$
\begin{aligned}
z &= \max\{c^T x \mid Ax = b, x \geq 0\} \\
w &= \min\{b^T y \mid A^T y \geq c\}
\end{aligned}
$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

5.2 Simplex and Duality

The following linear programs form a primal dual pair:

$$z = \max\{c^T x \mid Ax = b, x \geq 0\}$$
$$w = \min\{b^T y \mid A^T y \geq c\}$$

This means for computing the dual of a standard form LP, we do not have non-negativity constraints for the dual variables.

Proof

Primal:

$$\max\{c^T x \mid Ax = b, x \geq 0\}$$

Proof

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$$
\begin{aligned}
&\max\{c^Tx \mid Ax = b, x \geq 0\} \\
&\qquad = \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}
\end{aligned}
$$

Proof

Primal:

$$
\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\} \\
&\qquad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\qquad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}
$$

Proof

Primal:

$$\max\{c^Tx \mid Ax = b, x \geq 0\}$$
$$= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\}$$
$$= \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}$$

Dual:

$$\min\{[b^T \; -b^T]y \mid [A^T \; -A^T]y \geq c, y \geq 0\}$$

Proof

Primal:

$$\begin{aligned}
&\max\{c^T x \mid Ax = b, x \geq 0\} \\
&\qquad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\qquad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}$$

Dual:

$$\begin{aligned}
&\min\{[b^T\ -b^T]y \mid [A^T\ -A^T]y \geq c, y \geq 0\} \\
&\qquad = \min\left\{[b^T\ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T\ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\}
\end{aligned}$$

Proof

Primal:

$$\begin{aligned}\max\{c^Tx \mid Ax = b, x \geq 0\} \\ &= \max\{c^Tx \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\ &= \max\{c^Tx \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}\end{aligned}$$

Dual:

$$\begin{aligned}\min\{[b^T\ -b^T]y \mid [A^T\ -A^T]y \geq c, y \geq 0\} \\ &= \min\left\{[b^T\ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \;\middle|\; [A^T\ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0\right\} \\ &= \min\left\{b^T \cdot (y^+ - y^-) \;\middle|\; A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0\right\}\end{aligned}$$

Proof

Primal:

$$\begin{aligned}
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&\qquad = \max\{c^T x \mid Ax \leq b, -Ax \leq -b, x \geq 0\} \\
&\qquad = \max\{c^T x \mid \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix}, x \geq 0\}
\end{aligned}$$

Dual:

$$\begin{aligned}
&\min\{[b^T \ -b^T] y \mid [A^T \ -A^T] y \geq c, y \geq 0\} \\
&\qquad = \min\left\{ [b^T \ -b^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \,\middle|\, [A^T \ -A^T] \cdot \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\
&\qquad = \min\left\{ b^T \cdot (y^+ - y^-) \,\middle|\, A^T \cdot (y^+ - y^-) \geq c, y^- \geq 0, y^+ \geq 0 \right\} \\
&\qquad = \min\left\{ b^T y' \,\middle|\, A^T y' \geq c \right\}
\end{aligned}$$

Proof of Optimality Criterion for Simplex

Suppose that we have a basic feasible solution with reduced cost

$$\tilde{c} = c^T - c_B^T A_B^{-1} A \le 0$$

This is equivalent to $A^T (A_B^{-1})^T c_B \ge c$

$y^* = (A_B^{-1})^T c_B$ is solution to the dual $\min\{b^T y | A^T y \ge c\}$.

Hence, the solution is optimal.

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$$\begin{aligned} b^T y^* &= (Ax^*)^T y^* = (A_B x_B^*)^T y^* \\ &= (A_B x_B^*)^T (A_B^{-1})^T c_B = (x_B^*)^T A_B^T (A_B^{-1})^T c_B \\ &= c^T x^* \end{aligned}$$

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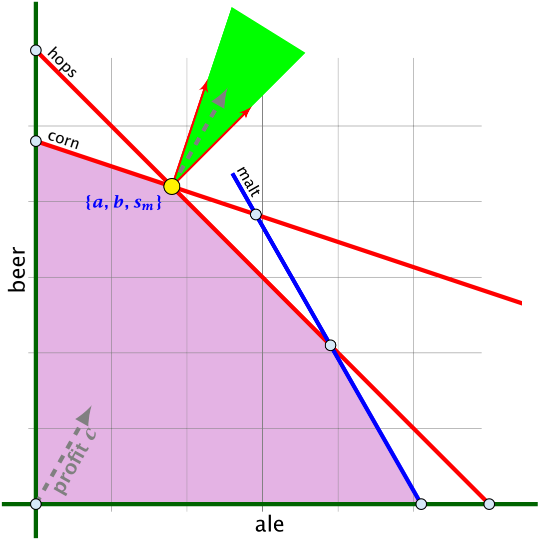
?



The profit vector c lies in the cone generated by the normals for the hops and the corn constraint.

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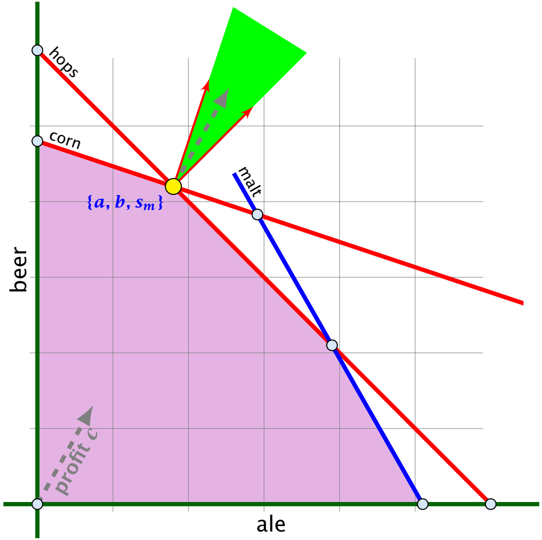

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Strong Duality

Theorem 30 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

$$z^* = w^*$$

Strong Duality

Theorem 31 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z^* and w^* denote the optimal solution to P and D, respectively. Then

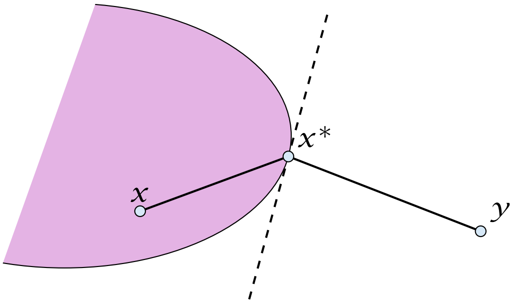
$$z^* = w^*$$

Lemma 32 (Weierstrass)

Let X be a compact set and let $f(x)$ be a continuous function on X. Then $\min\{f(x) : x \in X\}$ exists.

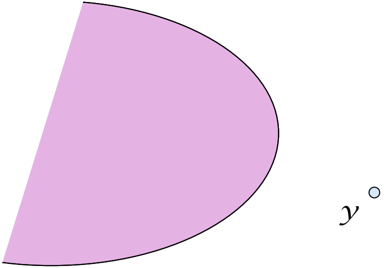
Lemma 33 (Projection Lemma)

Let $X \subseteq \mathbb{R}^m$ be a non-empty convex set, and let $y \notin X$. Then there exist $x^* \in X$ with minimum distance from y. Moreover for all $x \in X$ we have $(y - x^*)^T(x - x^*) \le 0$.



Proof of the Projection Lemma

- Define $f(x) = \|y - x\|$.
- We want to apply Weierstrass but X may not be bounded.
- $X \neq \emptyset$. Hence, there exists $x' \in X$.
- Define $X' = \{x \in X \mid \|y - x\| \leq \|y - x'\|\}$. This set is closed and bounded.
- Applying Weierstrass gives the existence.



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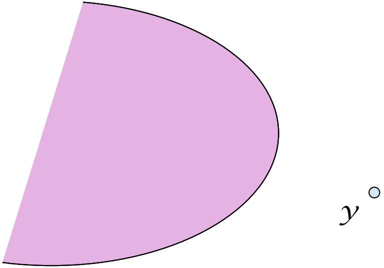

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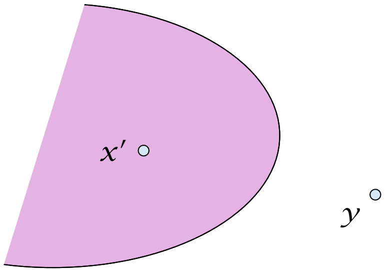

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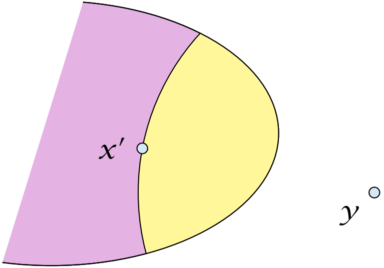

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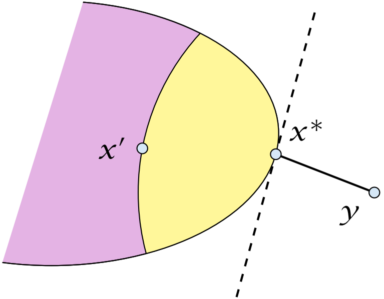

Proof of the Projection Lemma (continued)

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x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \le \epsilon \le 1$.

$$\|y - x^*\|^2$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \le \epsilon \le 1$.

$$\|y - x^*\|^2 \le \|y - x^* - \epsilon(x - x^*)\|^2$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$\begin{aligned}\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \le \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \le \epsilon \le 1$.

$$\begin{aligned}\|y - x^*\|^2 &\le \|y - x^* - \epsilon(x - x^*)\|^2 \\ &= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)\end{aligned}$$

Hence, $(y - x^*)^T(x - x^*) \le \frac{1}{2}\epsilon\|x - x^*\|^2$.

Proof of the Projection Lemma (continued)

x^* is minimum. Hence $\|y - x^*\|^2 \leq \|y - x\|^2$ for all $x \in X$.

By convexity: $x \in X$ then $x^* + \epsilon(x - x^*) \in X$ for all $0 \leq \epsilon \leq 1$.

$$
\begin{aligned}
\|y - x^*\|^2 &\leq \|y - x^* - \epsilon(x - x^*)\|^2 \\
&= \|y - x^*\|^2 + \epsilon^2\|x - x^*\|^2 - 2\epsilon(y - x^*)^T(x - x^*)
\end{aligned}
$$

Hence, $(y - x^*)^T(x - x^*) \leq \frac{1}{2}\epsilon\|x - x^*\|^2$.

Letting $\epsilon \to 0$ gives the result.

Theorem 34 (Separating Hyperplane)

Let $X \subseteq \mathbb{R}^m$ be a non-empty closed convex set, and let $y \notin X$. Then there exists a separating hyperplane $\{x \in \mathbb{R} : a^T x = \alpha\}$ where $a \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$ that separates y from X. ($a^T y < \alpha$; $a^T x \geq \alpha$ for all $x \in X$)

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
- Choose $a = (x^* - y)$ and $\alpha = a^T x^*$.
- For $x \in X : a^T(x - x^*) \geq 0$, and, hence, $a^T x \geq \alpha$.
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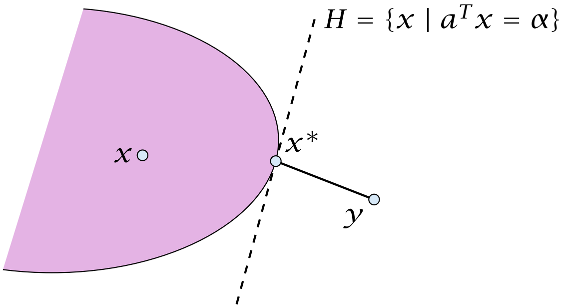

Proof of the Hyperplane Lemma

- Let $x^* \in X$ be closest point to y in X.
- By previous lemma $(y - x^*)^T(x - x^*) \leq 0$ for all $x \in X$.
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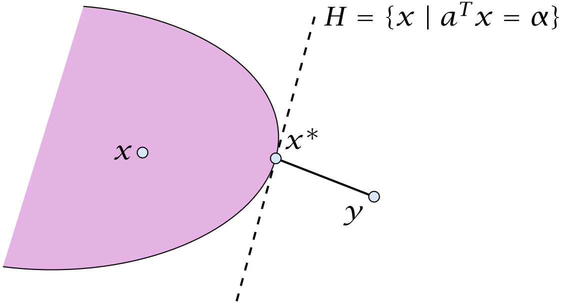

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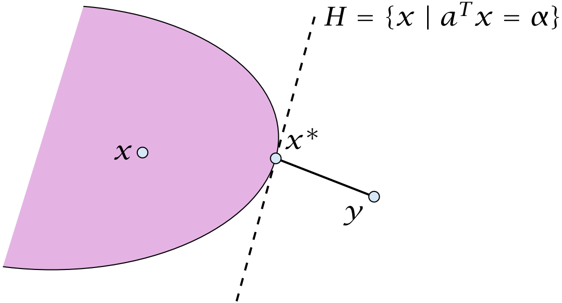

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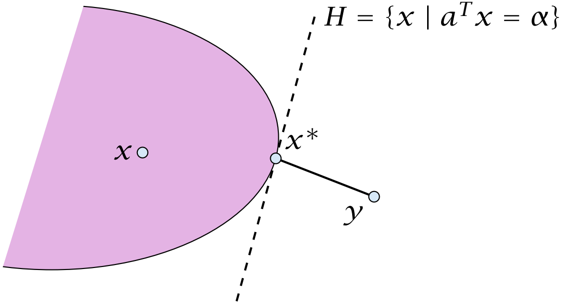

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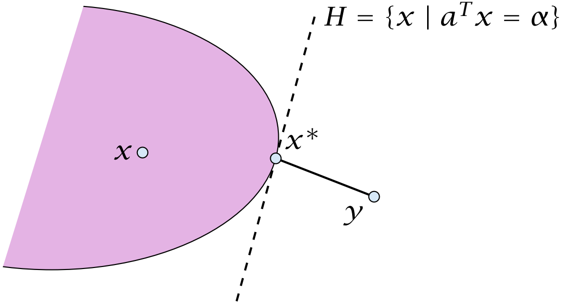

Lemma 35 (Farkas Lemma)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

1. $\exists x \in \mathbb{R}^n$ with $Ax = b$, $x \geq 0$
2. $\exists y \in \mathbb{R}^m$ with $A^T y \geq 0$, $b^T y < 0$

Assume $\hat{x}$ satisfies 1. and $\hat{y}$ satisfies 2. Then

$$0 > y^T b = y^T A x \geq 0$$

Hence, at most one of the statements can hold.

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Hence, at most one of the statements can hold.

Proof of Farkas Lemma

Now, assume that 1. does not hold.

Consider $S = \{Ax : x \geq 0\}$ so that S closed, convex, $b \notin S$.

We want to show that there is y with $A^T y \geq 0$, $b^T y < 0$.

Let y be a hyperplane that separates b from S. Hence, $y^T b < \alpha$ and $y^T s \geq \alpha$ for all $s \in S$.

$0 \in S \Rightarrow \alpha \leq 0 \Rightarrow y^T b < 0$

$y^T Ax \geq \alpha$ for all $x \geq 0$. Hence, $y^T A \geq 0$ as we can choose x arbitrarily large.

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Lemma 36 (Farkas Lemma; different version)

Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then exactly one of the following statements holds.

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Rewrite the conditions:

1. $\exists x \in \mathbb{R}^n$ with $\begin{bmatrix} A & I \end{bmatrix} \cdot \begin{bmatrix} x \\ s \end{bmatrix} = b$, $x \ge 0$, $s \ge 0$
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Proof of Strong Duality

P: $z = \max\{c^T x \mid Ax \le b, x \ge 0\}$

D: $w = \min\{b^T y \mid A^T y \ge c, y \ge 0\}$

Theorem 37 (Strong Duality)

Let P and D be a primal dual pair of linear programs, and let z and w denote the optimal solution to P and D, respectively (i.e., P and D are non-empty). Then

$$z = w \ .$$

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$$\begin{array}{lrcl} \exists x \in \mathbb{R}^n & & & \\ \text{s.t.} & Ax & \leq & b \\ & -c^T x & \leq & -\alpha \\ & x & \geq & 0 \end{array}$$

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From the definition of α we know that the first system is infeasible; hence the second must be feasible.

Proof of Strong Duality

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If the solution y, v has $v = 0$ we have that

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is feasible. By Farkas lemma this gives that LP P is infeasible. Contradiction to the assumption of the lemma.

Proof of Strong Duality

Hence, there exists a solution y, v with $v > 0$.

We can rescale this solution (scaling both y and v) s.t. $v = 1$.

Then y is feasible for the dual but $b^T y < \alpha$. This means that $w < \alpha$.

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Fundamental Questions

Definition 38 (Linear Programming Problem (LP))

Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, $\alpha \in \mathbb{Q}$. Does there exist $x \in \mathbb{Q}^n$ s.t. $Ax = b$, $x \geq 0$, $c^T x \geq \alpha$?

Questions:

- Is LP in NP?
- Is LP in co-NP? yes!
- Is LP in P?

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Complementary Slackness

Lemma 39

Assume a linear program $P = \max\{c^T x \mid Ax \le b; x \ge 0\}$ *has solution* x^* *and its dual* $D = \min\{b^T y \mid A^T y \ge c; y \ge 0\}$ *has solution* y^*.

1. *If* $x_j^* > 0$ *then the* j*-th constraint in* D *is tight.*
2. *If the* j*-th constraint in* D *is not tight than* $x_j^* = 0$.
3. *If* $y_i^* > 0$ *then the* i*-th constraint in* P *is tight.*
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If we say that a variable x_j^* (y_i^*) has slack if $x_j^* > 0$ ($y_i^* > 0$), (i.e., the corresponding variable restriction is not tight) and a contraint has slack if it is not tight, then the above says that for a primal-dual solution pair it is not possible that a constraint **and** its corresponding (dual) variable has slack.

Proof: Complementary Slackness

Analogous to the proof of weak duality we obtain

$$c^T x^* \le y^{*T} A x^* \le b^T y^*$$

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From the constraint of the dual it follows that $y^T A \ge c^T$. Hence the left hand side is a sum over the product of non-negative numbers. Hence, if e.g. $(y^T A - c^T)_j > 0$ (the j-th constraint in the dual is not tight) then $x_j = 0$ (2.). The result for (1./3./4.) follows similarly.

Interpretation of Dual Variables

- Brewer: find mix of ale and beer that maximizes profits

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\begin{array}{rrcrl}
\max & 13a & + & 23b & \\
\text{s.t.} & 5a & + & 15b & \le 480 \\
 & 4a & + & 4b & \le 160 \\
 & 35a & + & 20b & \le 1190 \\
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- Entrepeneur: buy resources from brewer at minimum cost
  C, H, M: unit price for corn, hops and malt.

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Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by ε_C, ε_H, and ε_M, respectively.

The profit increases to $\max\{c^Tx \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

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$$\begin{array}{lrcl} \min & (b^T + \epsilon^T)y & & \\ \text{s.t.} & A^T y & \ge & c \\ & y & \ge & 0 \end{array}$$

Interpretation of Dual Variables

Marginal Price:

- How much money is the brewer willing to pay for additional amount of Corn, Hops, or Malt?
- We are interested in the marginal price, i.e., what happens if we increase the amount of Corn, Hops, and Malt by $\varepsilon_C, \varepsilon_H$, and ε_M, respectively.

The profit increases to $\max\{c^T x \mid Ax \le b + \varepsilon; x \ge 0\}$. Because of strong duality this is equal to

$$\begin{array}{lrcl} \min & (b^T + \epsilon^T)y & & \\ \text{s.t.} & A^T y & \ge & c \\ & y & \ge & 0 \end{array}$$

Interpretation of Dual Variables

If ϵ is "small" enough then the optimum dual solution y^* might not change. Therefore the profit increases by $\sum_i \varepsilon_i y_i^*$.

Therefore we can interpret the dual variables as marginal prices.

Note that with this interpretation, complementary slackness becomes obvious.

- If the brewer has slack of some resource (e.g. corn) then he is not willing to pay anything for it (corresponding dual variable is zero).
- If the dual variable for some resource is non-zero, then an increase of this resource increases the profit of the brewer. Hence, it makes no sense to have left-overs of this resource. Therefore its slack must be zero.

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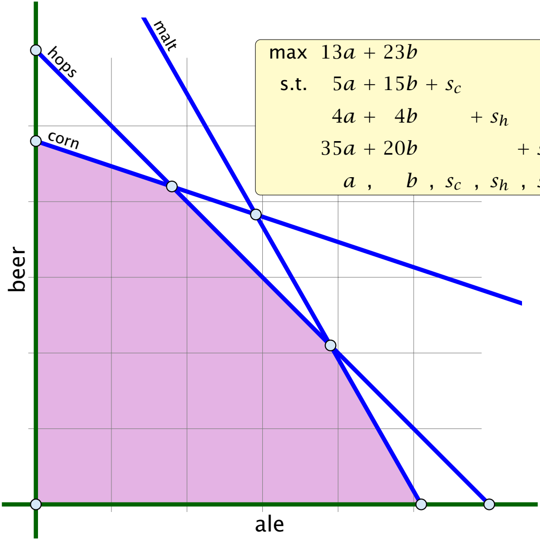
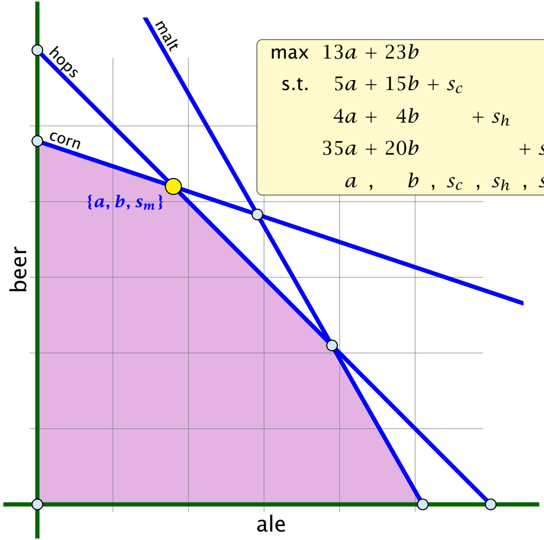
Example



$$
\begin{array}{lrcrcccl}
\max & 13a & + & 23b & & & & \\
\text{s.t.} & 5a & + & 15b & + s_c & & & = 480 \\
 & 4a & + & 4b & & + s_h & & = 160 \\
 & 35a & + & 20b & & & + s_m & = 1190 \\
 & a & , & b & , s_c & , s_h & , s_m & \geq 0
\end{array}
$$

Example



$$
\begin{array}{llllllll}
\text{max} & 13a + 23b & & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
& 4a + 4b & & + s_h & & = & 160 \\
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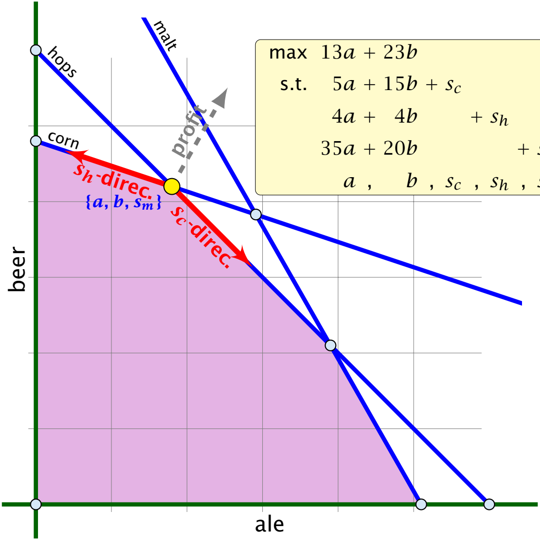
Example



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\max & 13a + 23b & & & & & \\
\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
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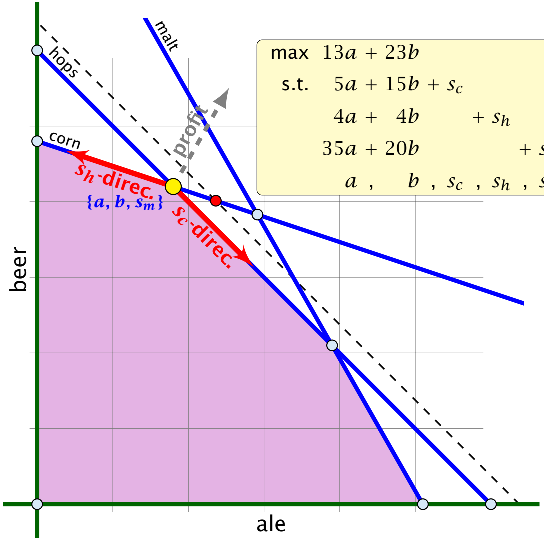
Example



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\text{s.t.} & 5a + 15b & + s_c & & & = & 480 \\
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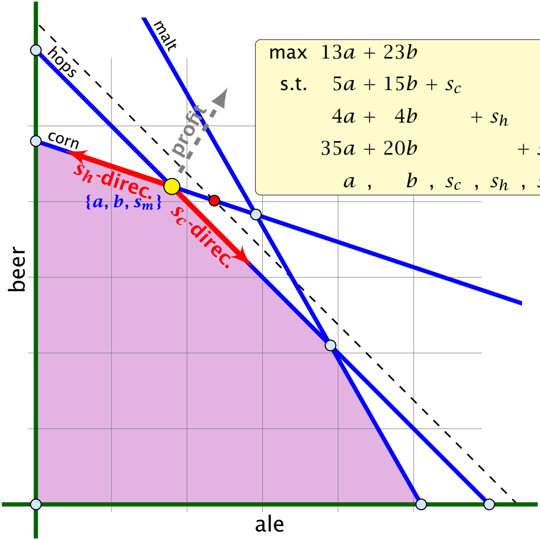
Example



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\begin{array}{llllllll}
\text{max} & 13a & + \; 23b & & & & & \\
\text{s.t.} & 5a & + \; 15b & + \; s_c & & & = & 480 \\
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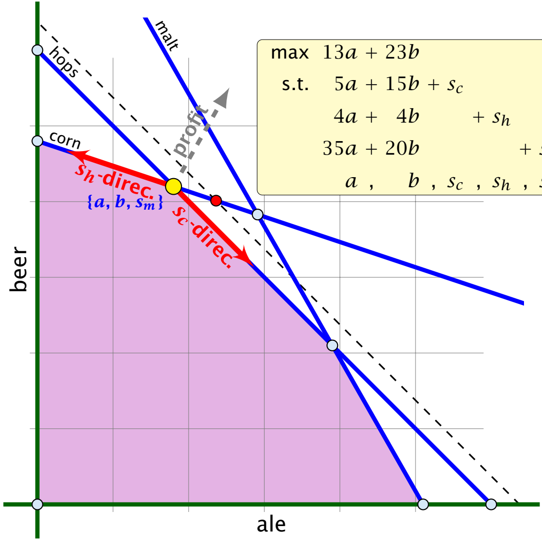

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\begin{array}{lrcrcrcrcrcl}
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\text{s.t.} & 5a & + & 15b & + & s_c & & & & & = & 480 \\
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\end{array}
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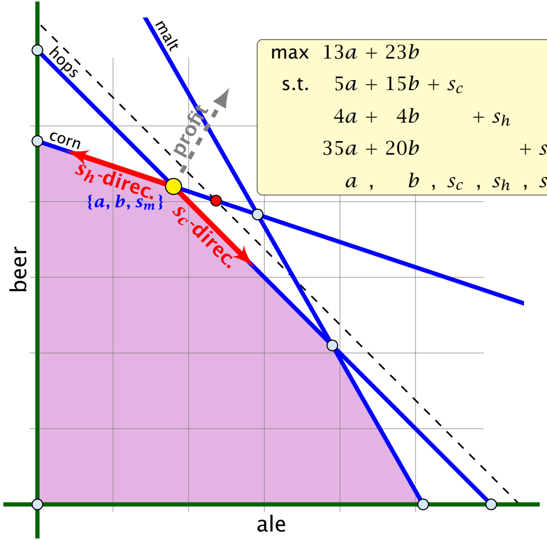
The change in profit when increasing hops by one unit is $= c_B^T A_B^{-1} e_h$.

Example



$$
\begin{array}{llcl}
\text{max} & 13a + 23b & & \\
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 & 35a + 20b + s_m & = & 1190 \\
 & a, b, s_c, s_h, s_m & \geq & 0
\end{array}
$$

The change in profit when increasing hops by one unit is

$= \underbrace{c_B^T A_B^{-1}}_{y^*} e_h$.

Of course, the previous argument about the increase in the primal objective only holds for the non-degenerate case.

If the optimum basis is degenerate then increasing the supply of one resource may not allow the objective value to increase.

Flows

Definition 40

An (s,t)-flow in a (complete) directed graph $G = (V, V \times V, c)$ is a function $f : V \times V \mapsto \mathbb{R}_0^+$ that satisfies

1. For each edge (x,y)

$$0 \le f_{xy} \le c_{xy} \ .$$

   (capacity constraints)

2. For each $v \in V \setminus \{s,t\}$

$$\sum_x f_{vx} = \sum_x f_{xv} \ .$$

   (flow conservation constraints)

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Flows

Definition 41
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \ .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

Flows

Definition 41
The value of an (s,t)-flow f is defined as

$$\mathrm{val}(f) = \sum_x f_{sx} - \sum_x f_{xs} \quad .$$

Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

LP-Formulation of Maxflow

$$
\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
 & \forall w \neq s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
 & & f_{zw} & \ge & 0 &
\end{array}
$$

$$
\begin{array}{lllcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
 & f_{st}: & 1\ell_{st} & \ge & 1 \\
 & f_{ts}: & 1\ell_{ts} & \ge & -1 \\
 & & \ell_{xy} & \ge & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - 1 + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + 1 & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - 0 + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \geq & 0 \\
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& f_{ty}\ (y \neq s,t): & 1\ell_{ty} - p_t + 1p_y & \geq & 0 \\
& f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + p_t & \geq & 0 \\
& f_{st}: & 1\ell_{st} - p_s + p_t & \geq & 0 \\
& f_{ts}: & 1\ell_{ts} - p_t + p_s & \geq & 0 \\
& & \ell_{xy} & \geq & 0
\end{array}
$$

with $p_t = 0$ and $p_s = 1$.

LP-Formulation of Maxflow

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\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & & \ell_{xy} & \geq & 0 \\
 & & p_s & = & 1 \\
 & & p_t & = & 0
\end{array}
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We can interpret the ℓ_{xy} value as assigning a length to every edge.

The value p_x for a variable, then can be seen as the distance of x to t (where the distance from s to t is required to be 1 since $p_s = 1$).

The constraint $p_x \leq \ell_{xy} + p_y$ then simply follows from triangle inequality ($d(x,t) \leq d(x,y) + d(y,t) \Rightarrow d(x,t) \leq \ell_{xy} + d(y,t)$).

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\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

This shows that the Maxflow/Mincut theorem follows from linear programming duality.

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Flows

Definition 43
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Maximum Flow Problem:
Find an (s,t)-flow with maximum value.

Flows

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Maximum Flow Problem:
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\begin{array}{llrcll}
\max & & \sum_z f_{sz} - \sum_z f_{zs} & & & \\
\text{s.t.} & \forall (z,w) \in V \times V & f_{zw} & \le & c_{zw} & \ell_{zw} \\
& \forall w \ne s,t & \sum_z f_{zw} - \sum_z f_{wz} & = & 0 & p_w \\
& & f_{zw} & \ge & 0 &
\end{array}
$$

$$
\begin{array}{lllcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \ne s,t): & 1\ell_{xy} - 1p_x + 1p_y & \ge & 0 \\
& f_{sy}\ (y \ne s,t): & 1\ell_{sy} \qquad + 1p_y & \ge & 1 \\
& f_{xs}\ (x \ne s,t): & 1\ell_{xs} - 1p_x & \ge & -1 \\
& f_{ty}\ (y \ne s,t): & 1\ell_{ty} \qquad + 1p_y & \ge & 0 \\
& f_{xt}\ (x \ne s,t): & 1\ell_{xt} - 1p_x & \ge & 0 \\
& f_{st}: & 1\ell_{st} & \ge & 1 \\
& f_{ts}: & 1\ell_{ts} & \ge & -1 \\
& & \ell_{xy} & \ge & 0
\end{array}
$$

LP-Formulation of Maxflow

$$
\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}\ (x,y \neq s,t): & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
 & f_{sy}\ (y \neq s,t): & 1\ell_{sy} - 1 + 1p_y & \geq & 0 \\
 & f_{xs}\ (x \neq s,t): & 1\ell_{xs} - 1p_x + 1 & \geq & 0 \\
 & f_{ty}\ (y \neq s,t): & 1\ell_{ty} - 0 + 1p_y & \geq & 0 \\
 & f_{xt}\ (x \neq s,t): & 1\ell_{xt} - 1p_x + 0 & \geq & 0 \\
 & f_{st}: & 1\ell_{st} - 1 + 0 & \geq & 0 \\
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LP-Formulation of Maxflow

$$\begin{array}{llrcl}
\min & & \sum_{(xy)} c_{xy}\ell_{xy} & & \\
\text{s.t.} & f_{xy}: & 1\ell_{xy} - 1p_x + 1p_y & \geq & 0 \\
& & \ell_{xy} & \geq & 0 \\
& & p_s & = & 1 \\
& & p_t & = & 0
\end{array}$$

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One can show that there is an optimum LP-solution for the dual problem that gives an integral assignment of variables.

This means $p_x = 1$ or $p_x = 0$ for our case. This gives rise to a cut in the graph with vertices having value 1 on one side and the other vertices on the other side. The objective function then evaluates the capacity of this cut.

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Degeneracy Revisited

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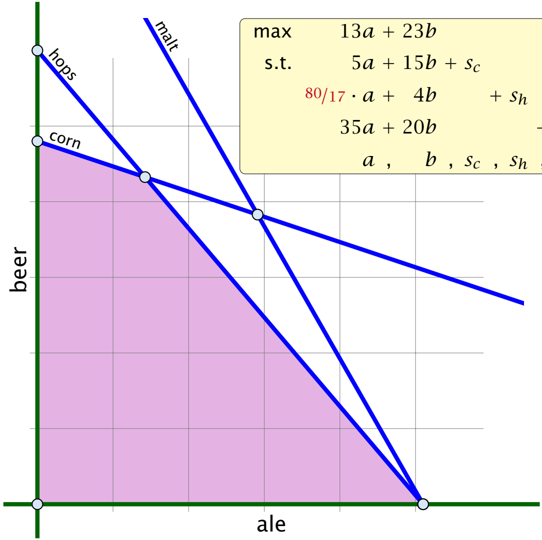
Degenerate Example



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\begin{array}{llllllll}
\text{max} & 13a & + 23b & & & & & \\
\text{s.t.} & 5a & + 15b & + s_c & & & = & 480 \\
 & {}^{80}/_{17} \cdot a & + \ \ 4b & & + s_h & & = & 160 \\
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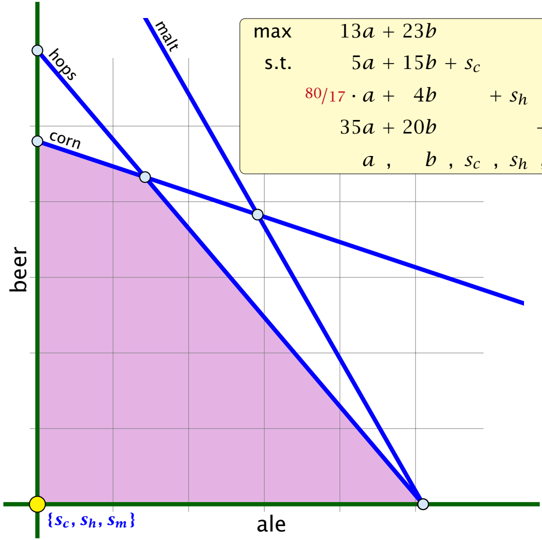

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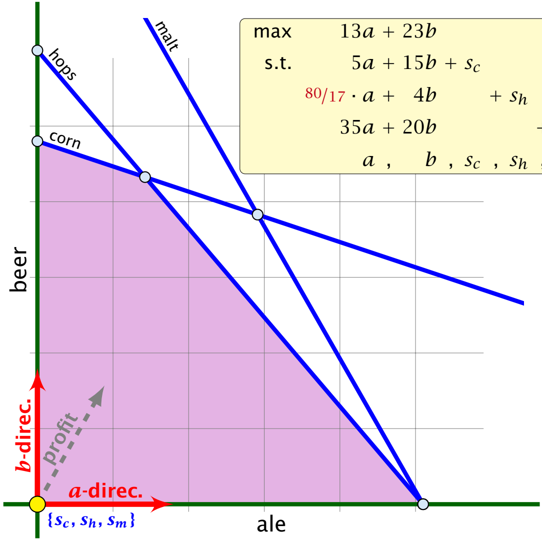

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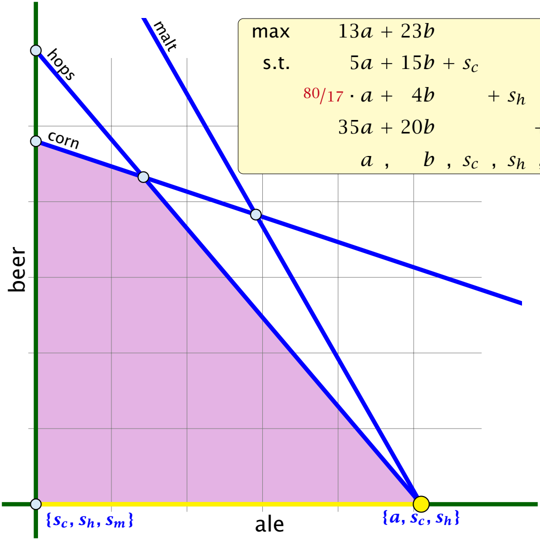

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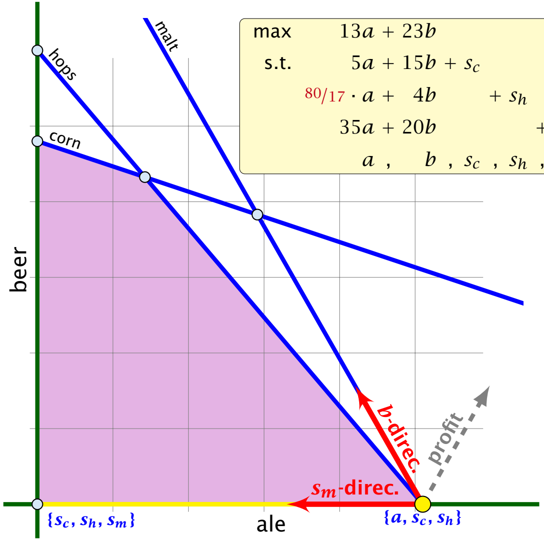

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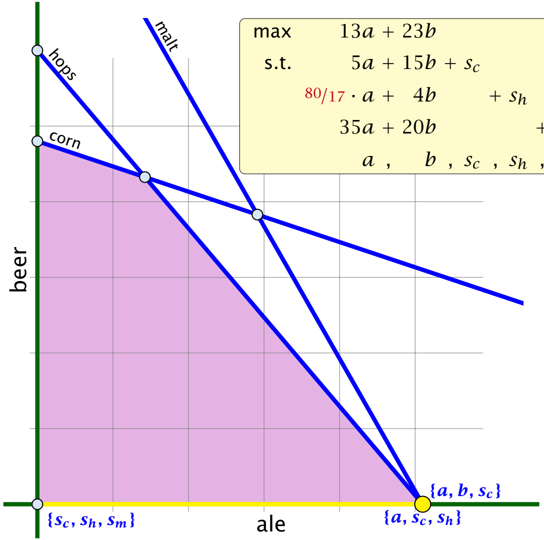

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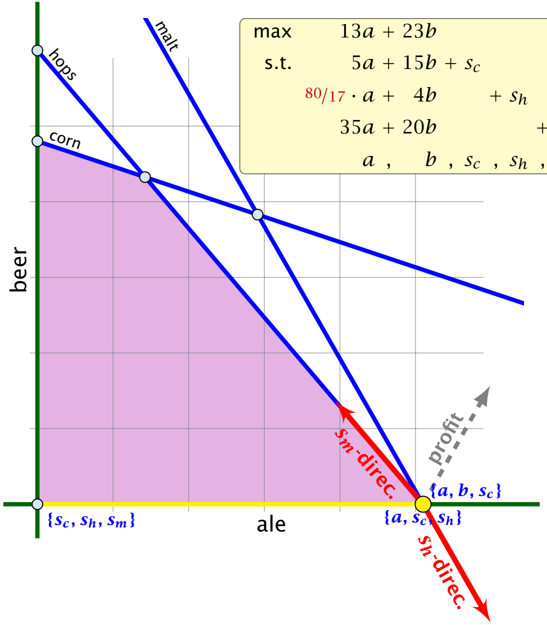

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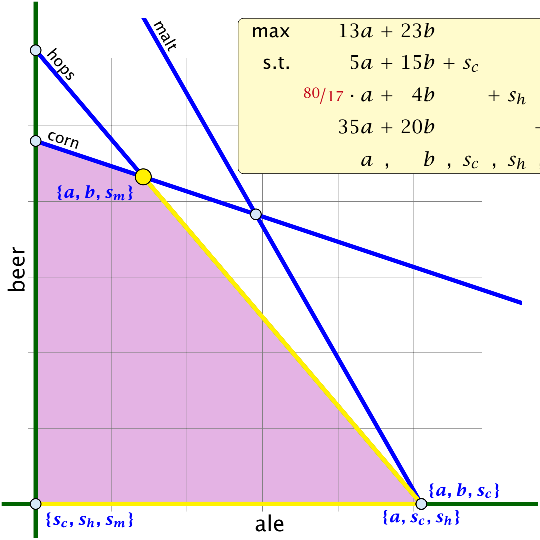

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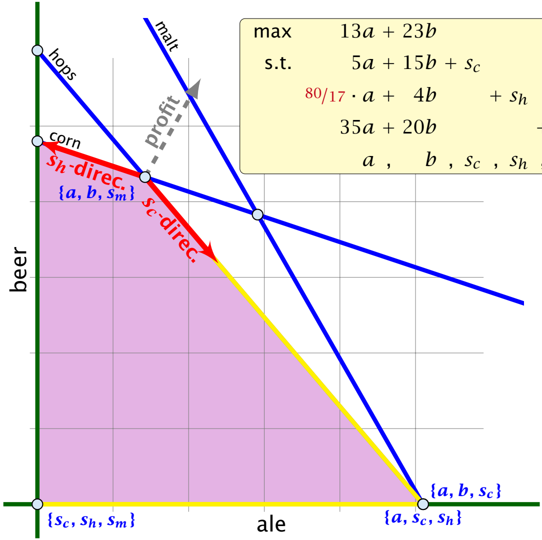

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- LP' is feasible
- If a set B of basis variables corresponds to an infeasible basis (i.e. $A_B^{-1} b \not\geq 0$) then B corresponds to an infeasible basis in LP' (note that columns in A_B are linearly independent).
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Perturbation

Let B be index set of some basis with basic solution

$$x_B^* = A_B^{-1}b \geq 0, x_N^* = 0 \quad \text{(i.e. } B \text{ is feasible)}$$

Fix

$$b' := b + A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix} \text{ for } \varepsilon > 0 \;.$$

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The new LP is feasible because the set B of basis variables provides a feasible basis:

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Hence, $\tilde{B}$ is not feasible.

Property III

Let $\tilde{B}$ be a basis. It has an associated solution

$$x^*_{\tilde{B}} = A_{\tilde{B}}^{-1}b + A_{\tilde{B}}^{-1}A_B \begin{pmatrix} \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}$$

in the perturbed instance.

We can view each component of the vector as a polynom with variable ε of degree at most m.

$A_{\tilde{B}}^{-1}A_B$ has rank m. Therefore no polynom is 0.

A polynom of degree at most m has at most m roots (Nullstellen).

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- If it terminates because it finds a variable x_j with $\tilde{c}_j > 0$ for which the j-th basis direction d, fulfills $d \geq 0$ we know that LP' is unbounded. The basis direction does not depend on b. Hence, we also know that LP is unbounded.

Lexicographic Pivoting

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Simulate behaviour of LP' without explicitly doing a perturbation.

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If we do not have a choice for the leaving variable then LP' and LP do the same (i.e., choose the same variable).

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In the following we assume that $b \geq 0$. This can be obtained by replacing the initial system $(A_B \mid b)$ by $(A_B^{-1}A \mid A_B^{-1}b)$ where B is the index set of a feasible basis (found e.g. by the first phase of the Two-phase algorithm).

Then the perturbed instance is

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Matrix View

Let our linear program be

$$\begin{array}{ccccc} c_B^T x_B & + & c_N^T x_N & = & Z \\ A_B x_B & + & A_N x_N & = & b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The simplex tableaux for basis B is

$$\begin{array}{ccccc} & & (c_N^T - c_B^T A_B^{-1} A_N) x_N & = & Z - c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ x_B & , & x_N & \geq & 0 \end{array}$$

The BFS is given by $x_N = 0, x_B = A_B^{-1} b$.

If $(c_N^T - c_B^T A_B^{-1} A_N) \leq 0$ we know that we have an optimum solution.

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LP chooses an arbitrary leaving variable that has $\hat{A}_{\ell e} > 0$ and minimizes

$$\theta_\ell = \frac{\hat{b}_\ell}{\hat{A}_{\ell e}} = \frac{(A_B^{-1}b)_\ell}{(A_B^{-1}A_{*e})_\ell} \enspace .$$

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Definition 44
$u \leq_{\text{lex}} v$ if and only if the first component in which u and v differ fulfills $u_i \leq v_i$.

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$$= \frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}\begin{pmatrix}1\\ \varepsilon\\ \vdots \\ \varepsilon^m\end{pmatrix}$$

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This means you can choose the variable/row ℓ for which the vector

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Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

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This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

Lexicographic Pivoting

This means you can choose the variable/row ℓ for which the vector

$$\frac{\ell\text{-th row of } A_B^{-1}(b \mid I)}{(A_B^{-1}A_{*e})_\ell}$$

is lexicographically minimal.

Of course only including rows with $(A_B^{-1}A_{*e})_\ell > 0$.

This technique guarantees that your pivoting is the same as in the perturbed case. This guarantees that cycling does not occur.

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Can we obtain a better analysis?

Number of Simplex Iterations

Observation

Simplex visits every feasible basis at most once.

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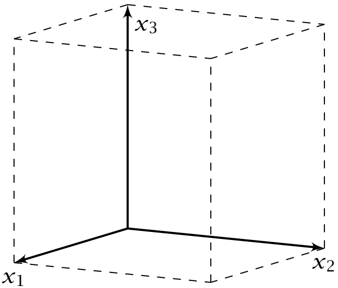
Simplex visits every feasible basis at most once.

However, also the number of feasible bases can be very large.

Example

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & 0 \le x_1 \le 1 \\ & 0 \le x_2 \le 1 \\ & \vdots \\ & 0 \le x_n \le 1 \end{aligned}$$



$2n$ constraint on n variables define an n-dimensional hypercube as feasible region.

The feasible region has 2^n vertices.

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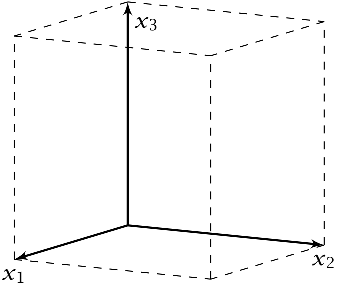


However, Simplex may still run quickly as it usually does not visit all feasible bases.

In the following we give an example of a feasible region for which there is a bad Pivoting Rule.

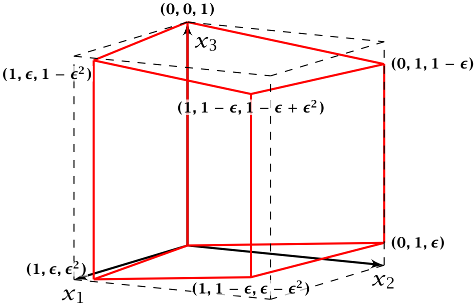
Pivoting Rule

A Pivoting Rule defines how to choose the entering and leaving variable for an iteration of Simplex.

In the non-degenerate case after choosing the entering variable the leaving variable is unique.

Klee Minty Cube

$$\begin{aligned} \max\ & x_n \\ \text{s.t.}\ & \quad 0 \le x_1 \le 1 \\ & \epsilon x_1 \le x_2 \le 1 - \epsilon x_1 \\ & \epsilon x_2 \le x_3 \le 1 - \epsilon x_2 \\ & \qquad \vdots \\ & \epsilon x_{n-1} \le x_n \le 1 - \epsilon x_{n-1} \\ & \quad x_i \ge 0 \end{aligned}$$



Observations

- We have $2n$ constraints, and $3n$ variables (after adding slack variables to every constraint).
- Every basis is defined by $2n$ variables, and n non-basic variables.
- There exist degenerate vertices.
- The degeneracies come from the non-negativity constraints, which are superfluous.
- In the following all variables x_i stay in the basis at all times.
- Then, we can uniquely specify a basis by choosing for each variable whether it should be equal to its lower bound, or equal to its upper bound (the slack variable corresponding to the non-tight constraint is part of the basis).
- We can also simply identify each basis/vertex with the corresponding hypercube vertex obtained by letting $\epsilon \to 0$.

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Analysis

- In the following we specify a sequence of bases (identified by the corresponding hypercube node) along which the objective function strictly increases.
- The basis $(0, \ldots, 0, 1)$ is the unique optimal basis.
- Our sequence S_n starts at $(0, \ldots, 0)$ ends with $(0, \ldots, 0, 1)$ and visits every node of the hypercube.
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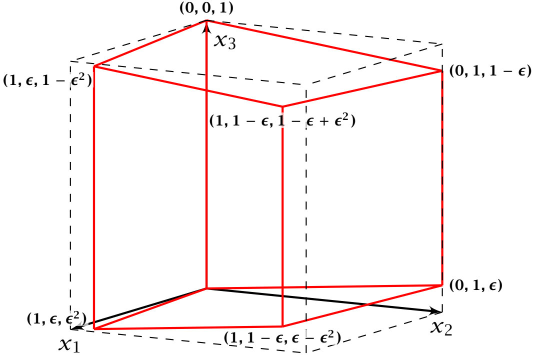

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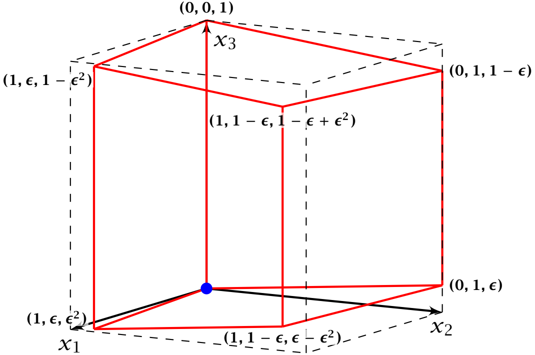

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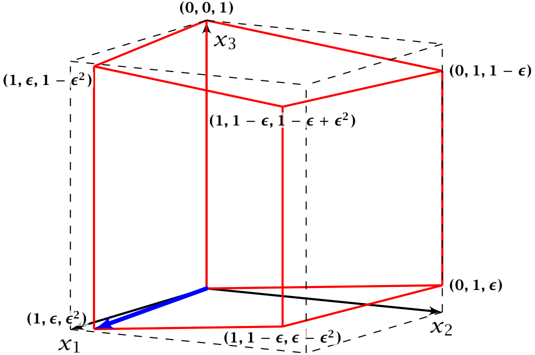

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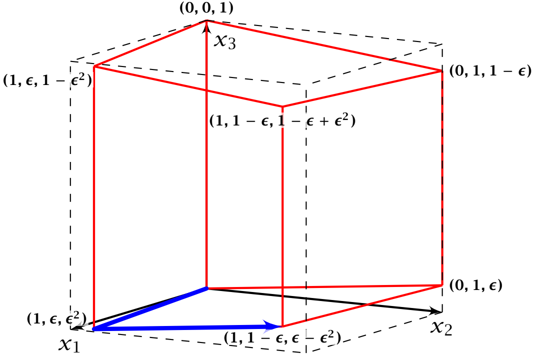

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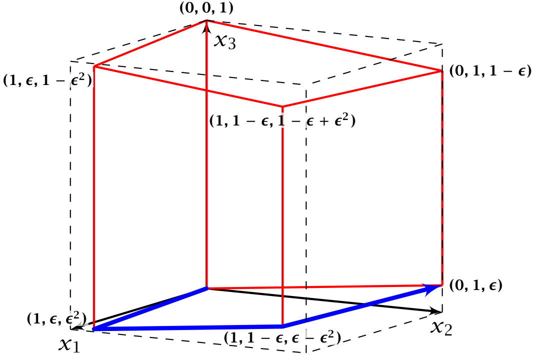

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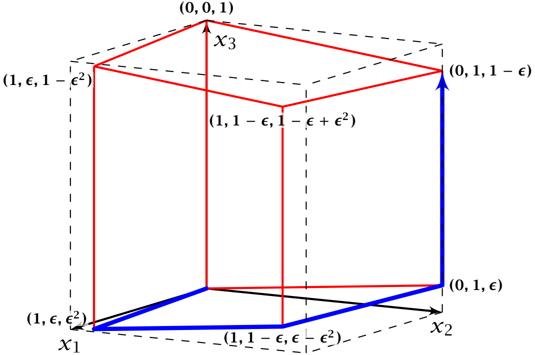

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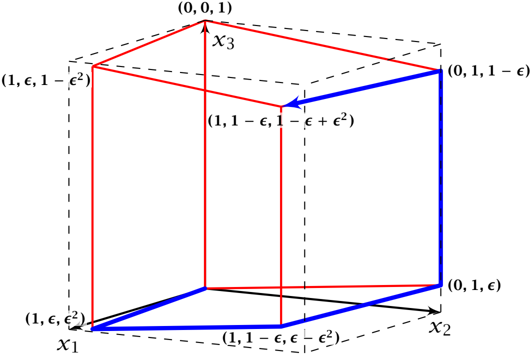

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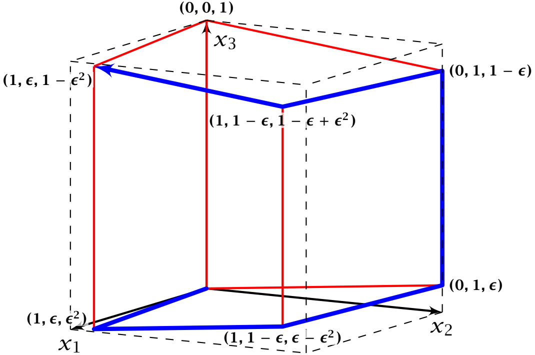

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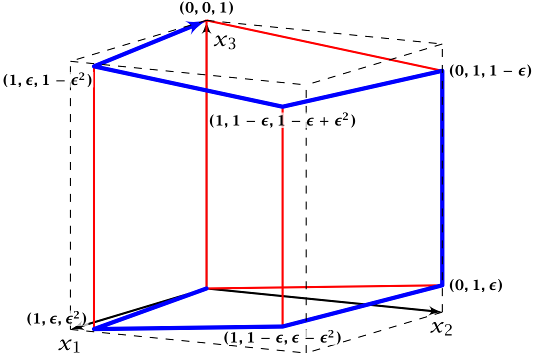

Analysis

The sequence S_n that visits every node of the hypercube is defined recursively

$$\begin{tikzcd}
(0,\dots,0,0,0) \arrow[d, rightsquigarrow, "S_{n-1}"] \arrow[ddd, rightsquigarrow, bend left, "S_n"] \\
(0,\dots,0,1,0) \arrow[d] \\
(0,\dots,0,1,1) \arrow[d, rightsquigarrow, "S^{\text{rev}}_{n-1}"] \\
(0,\dots,0,0,1)
\end{tikzcd}$$

The non-recursive case is $S_1 = 0 \to 1$

Analysis

Lemma 45

The objective value x_n is increasing along path S_n.

Proof by induction:

$n = 1$: obvious, since $S_1 = 0 \to 1$, and $1 > 0$.

$n - 1 \to n$

- For the first part the value of $x_n = \epsilon x_{n-1}$.
- By induction hypothesis x_{n-1} is increasing along S_{n-1}, hence, also x_n.
- Going from $(0, \dots, 0, 1, 0)$ to $(0, \dots, 0, 1, 1)$ increases x_n for small enough ϵ.
- For the remaining path S_{n-1}^{rev} we have $x_n = 1 - \epsilon x_{n-1}$.
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Remarks about Simplex

Observation

The simplex algorithm takes at most $\binom{n}{m}$ iterations. Each iteration can be implemented in time $\mathcal{O}(mn)$.

In practise it usually takes a linear number of iterations.

Remarks about Simplex

Theorem

For almost all known deterministic pivoting rules (rules for choosing entering and leaving variables) there exist lower bounds that require the algorithm to have exponential running time ($\Omega(2^{\Omega(n)})$) (e.g. Klee Minty 1972).

Remarks about Simplex

Theorem
For some standard randomized pivoting rules there exist subexponential lower bounds ($\Omega(2^{\Omega(n^{\alpha})})$ for $\alpha > 0$) (Friedmann, Hansen, Zwick 2011).

Remarks about Simplex

Conjecture (Hirsch 1957)
The edge-vertex graph of an m-facet polytope in d-dimensional Euclidean space has diameter no more than $m - d$.

The conjecture has been proven wrong in 2010.

But the question whether the diameter is perhaps of the form $\mathcal{O}(\mathrm{poly}(m, d))$ is open.

8 Seidels LP-algorithm

- Suppose we want to solve $\min\{c^T x \mid Ax \geq b; x \geq 0\}$, where $x \in \mathbb{R}^d$ and we have m constraints.
- In the worst-case Simplex runs in time roughly $\mathcal{O}(m(m+d)\binom{m+d}{m}) \approx (m+d)^m$. (slightly better bounds on the running time exist, but will not be discussed here).
- If d is much smaller than m one can do a lot better.
- In the following we develop an algorithm with running time $\mathcal{O}(d! \cdot m)$, i.e., linear in m.

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Setting:

- We assume an LP of the form

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

- We assume that the LP is bounded.

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{lrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.**

Computing a Lower Bound

Let s denote the smallest common multiple of all denominators of entries in A, b.

Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

Computing a Lower Bound

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Multiply entries in A, b by s to obtain integral entries. This does not change the feasible region.

Add slack variables to A; denote the resulting matrix with $\bar{A}$.

If B is an optimal basis then x_B with $\bar{A}_B x_B = \bar{b}$, gives an optimal assignment to the basis variables (non-basic variables are 0).

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Theorem 46 (Cramers Rule)
Let M be a matrix with $\det(M) \neq 0$. Then the solution to the system $Mx = b$ is given by

$$x_j = \frac{\det(M_j)}{\det(M)} ,$$

where M_j is the matrix obtained from M by replacing the j-th column by the vector b.

Proof:

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- Define

$$X_j = \begin{pmatrix} | & & | & | & | & & | \\ e_1 & \cdots & e_{j-1} & x & e_{j+1} & \cdots & e_n \\ | & & | & | & | & & | \end{pmatrix}$$

  Note that expanding along the j-th column gives that $\det(X_j) = x_j$.

- Further, we have

$$MX_j = \begin{pmatrix} | & & | & | & | & & | \\ Me_1 & \cdots & Me_{j-1} & Mx & Me_{j+1} & \cdots & Me_n \\ | & & | & | & | & & | \end{pmatrix} = M_j$$

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Bounding the Determinant

Let Z be the maximum absolute entry occuring in $\bar{A}, \bar{b}$ or c. Let C denote the matrix obtained from $\bar{A}_B$ by replacing the j-th column with vector $\bar{b}$.

Observe that

$$|\det(C)|$$

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Alternatively, Hadamards inequality gives

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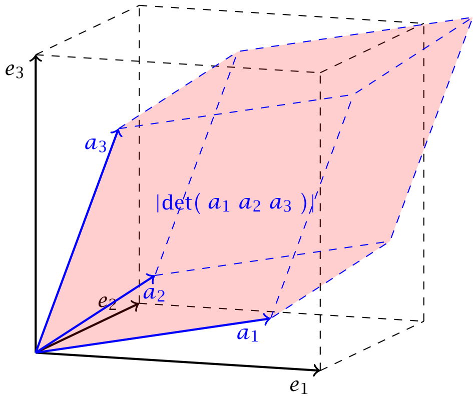
$$
\begin{aligned}
|\det(C)| &\le \prod_{i=1}^{m} \|C_{*i}\| \le \prod_{i=1}^{m} (\sqrt{m} Z) \\
&\le m^{m/2} Z^m \enspace .
\end{aligned}
$$

Hadamards Inequality



Hadamards inequality says that the volume of the red parallelepiped (Spat) is smaller than the volume in the black cube (if $\|e_1\| = \|a_1\|$, $\|e_2\| = \|a_2\|$, $\|e_3\| = \|a_3\|$).

Ensuring Conditions

Given a standard minimization LP

$$\begin{array}{rrcl} \min & c^T x & & \\ \text{s.t.} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

how can we obtain an LP of the required form?

- **Compute a lower bound on $c^T x$ for any basic feasible solution.** Add the constraint $c^T x \geq -mZ(m! \cdot Z^m) - 1$. Note that this constraint is superfluous unless the LP is unbounded.

Ensuring Conditions

Compute an optimum basis for the new LP.

- If the cost is $c^T x = -(mZ)(m! \cdot Z^m) - 1$ we know that the original LP is unbounded.
- Otw. we have an optimum basis.

In the following we use $\mathcal{H}$ to denote the set of all constraints apart from the constraint $c^T x \geq -mZ(m! \cdot Z^m) - 1$.

We give a routine SeidelLP($\mathcal{H}, d$) that is given a set $\mathcal{H}$ of explicit, non-degenerate constraints over d variables, and minimizes $c^T x$ over all feasible points.

In addition it obeys the implicit constraint $c^T x \geq -(mZ)(m! \cdot Z^m) - 1$.

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if $d = 1$ **then** solve 1-dimensional problem and return;

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if $\hat{x}^* =$ infeasible **then**
    return infeasible
else
    add the value of x_ℓ to $\hat{x}^*$ and return the solution

8 Seidels LP-algorithm

- If $d = 1$ we can solve the 1-dimensional problem in time $\mathcal{O}(m)$.
- If $d > 1$ and $m = 0$ we take time $\mathcal{O}(d)$ to return d-dimensional vector x.
- The first recursive call takes time $T(m-1, d)$ for the call plus $\mathcal{O}(d)$ for checking whether the solution fulfills h.
- If we are unlucky and $\hat{x}^*$ does not fulfill h we need time $\mathcal{O}(d(m+1)) = \mathcal{O}(dm)$ to eliminate x_ℓ. Then we make a recursive call that takes time $T(m-1, d-1)$.
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This gives the recurrence

$$T(m,d) = \begin{cases} \mathcal{O}(m) & \text{if } d = 1 \\ \mathcal{O}(d) & \text{if } d > 1 \text{ and } m = 0 \\ \mathcal{O}(d) + T(m-1,d) + & \\ \frac{d}{m}(\mathcal{O}(dm) + T(m-1,d-1)) & \text{otw.} \end{cases}$$

Note that $T(m,d)$ denotes the expected running time.

8 Seidels LP-algorithm

Let C be the largest constant in the $\mathcal{O}$-notations.

$$T(m,d) = \begin{cases} Cm & \text{if } d = 1 \\ Cd & \text{if } d > 1 \text{ and } m = 0 \\ Cd + T(m-1,d) + & \\ \frac{d}{m}(Cdm + T(m-1,d-1)) & \text{otw.} \end{cases}$$

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(by induction hypothesis statm. true for $d' < d, m' \geq 0$;
and for $d' = d$, $m' < m$)

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\begin{aligned}
T(m,d) &= \mathcal{O}(d) + T(m-1,d) + \frac{d}{m}\Big(\mathcal{O}(dm) + T(m-1,d-1)\Big) \\
&\leq Cd + Cf(d)(m-1) + Cd^2 + \frac{d}{m}Cf(d-1)(m-1) \\
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since $\sum_{i\geq 1} \frac{i^2}{i!}$ is a constant.

Complexity

LP Feasibility Problem (LP feasibility)

- Given $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. Does there exist $x \in \mathbb{R}$ with $Ax = b$, $x \geq 0$?
- Note that allowing A, b to contain rational numbers does not make a difference, as we can multiply every number by a suitable large constant so that everything becomes integral but the feasible region does not change.

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The Bit Model

Input size

- The number of bits to represent a number $a \in \mathbb{Z}$ is

$$\lceil \log_2(|a|) \rceil + 1$$

- Let for an $m \times n$ matrix M, $L(M)$ denote the number of bits required to encode all the numbers in M.

$$L(M) := \sum_{i,j} \lceil \log_2(|m_{ij}|) + 1 \rceil$$

- In the following we assume that input matrices are encoded in a standard way, where each number is encoded in binary and then suitable separators are added in order to separate distinct number from each other.
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- In the following we sometimes refer to $L := L([A|b])$ as the input size (even though the real input size is something in $\Theta(L([A|b]))$).
- In order to show that LP-decision is in NP we show that if there is a solution x then there exists a small solution for which feasibility can be verified in polynomial time (polynomial in $L([A|b])$).

Suppose that $Ax = b;\ x \geq 0$ is feasible.

Then there exists a basic feasible solution. This means a set B of basic variables such that

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Size of a Basic Feasible Solution

Lemma 47

Let $M \in \mathbb{Z}^{m\times m}$ be an invertable matrix and let $b \in \mathbb{Z}^m$. Further define $L' = L([M \mid b]) + n\log_2 n$. Then a solution to $Mx = b$ has rational components x_j of the form $\frac{D_j}{D}$, where $|D_j| \le 2^{L'}$ and $|D| \le 2^{L'}$.

Proof:

Cramers rules says that we can compute x_j as

$$x_j = \frac{\det(M_j)}{\det(M)}$$

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Analogously for $\det(M_j)$.

This means if $Ax = b, x \geq 0$ is feasible we only need to consider vectors x where an entry x_j can be represented by a rational number with encoding length polynomial in the input length L.

Hence, the x that we have to guess is of length polynomial in the input-length L.

For a given vector x of polynomial length we can check for feasibility in polynomial time.

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Reducing LP-solving to LP decision.

Given an LP $\max\{c^Tx \mid Ax = b; x \geq 0\}$ do a binary search for the optimum solution

(Add constraint $c^Tx - \delta = M$; $\delta \geq 0$ or ($c^Tx \geq M$). Then checking for feasibility shows whether optimum solution is larger or smaller than M).

If the LP is feasible then the binary search finishes in at most

$$\log_2\left(\frac{2n2^{2L'}}{1/2^{L'}}\right) = \mathcal{O}(L') \enspace ,$$

as the range of the search is at most $-n2^{2L'}, \dots, n2^{2L'}$ and the distance between two adjacent values is at least $\frac{1}{\det(A)} \geq \frac{1}{2^{L'}}$.

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How do we detect whether the LP is unbounded?

Let $M_{\max} = n2^{2L'}$ be an upper bound on the objective value of a basic feasible solution.

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Ellipsoid Method

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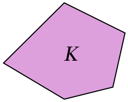
- Let K be a convex set.



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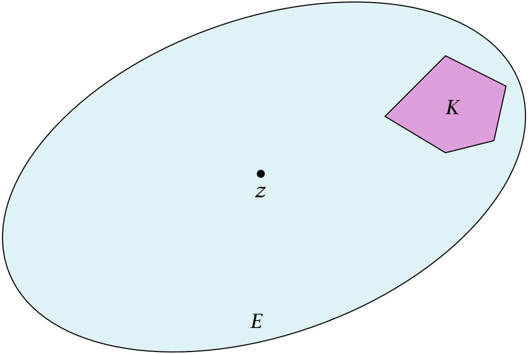

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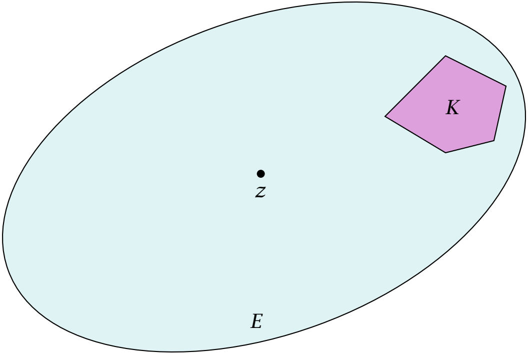

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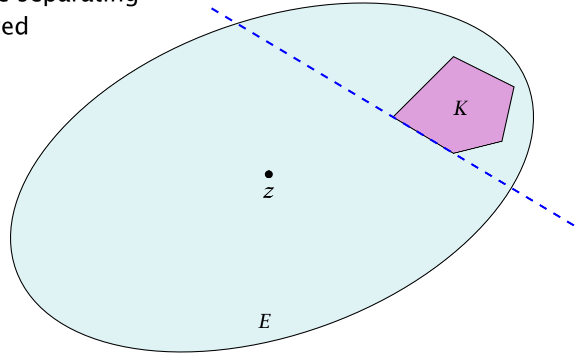

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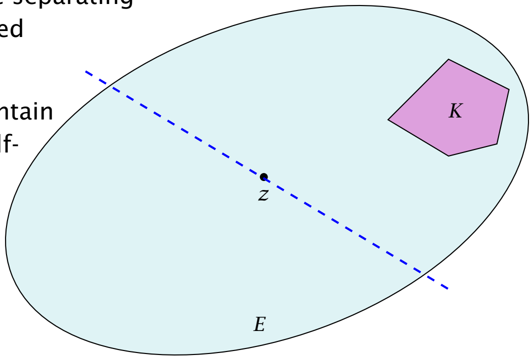

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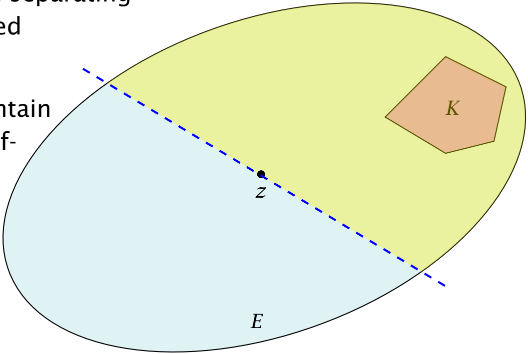

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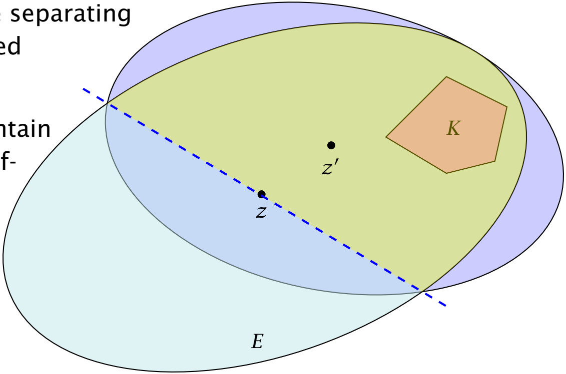

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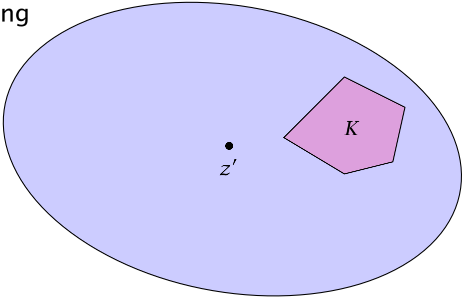

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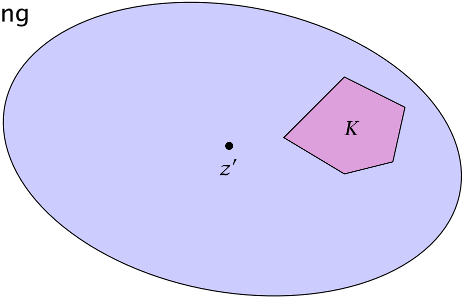
- Otw. find a hyperplane separating K from z (e.g. a violated constraint in the LP).
- Shift hyperplane to contain node z. H denotes half-space that contains K.
- Compute (smallest) ellipsoid E' that contains $K \cap H$.
- REPEAT



Issues/Questions:

- How do you choose the first Ellipsoid? What is its volume?
- What if the polytop K is unbounded?
- How do you measure progress? By how much does the volume decrease in each iteration?
- When can you stop? What is the minimum volume of a non-empty polytop?

Definition 48

A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(x) = Lx + t$, where L is an invertible matrix is called an affine transformation.

Definition 49

A ball in $\mathbb{R}^n$ with center c and radius r is given by

$$\begin{aligned} B(c,r) &= \{x \mid (x-c)^T(x-c) \le r^2\} \\ &= \{x \mid \sum_i (x-c)_i^2/r^2 \le 1\} \end{aligned}$$

$B(0,1)$ is called the unit ball.

Definition 50
An affine transformation of the unit ball is called an ellipsoid.

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f(B(0,1)) &= \{f(x) \mid x \in B(0,1)\} \\
&= \{y \in \mathbb{R}^n \mid L^{-1}(y - t) \in B(0,1)\}
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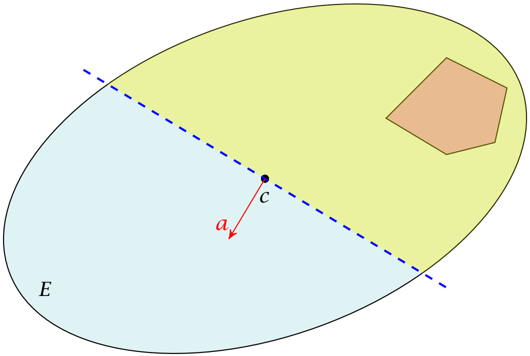
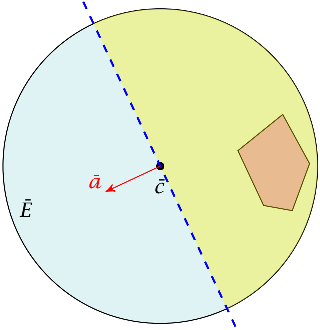
where $Q = LL^T$ is an invertible matrix.

How to Compute the New Ellipsoid



How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



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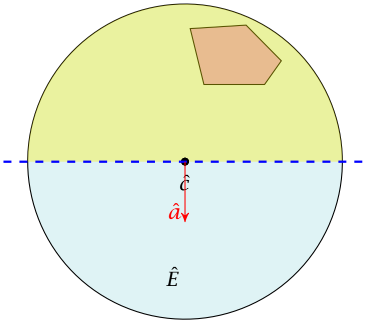
- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.
- Use a rotation R^{-1} to rotate the unit ball such that the normal vector of the halfspace is parallel to e_1.



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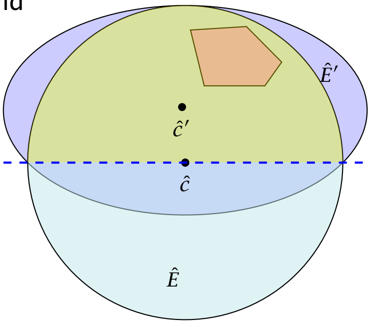

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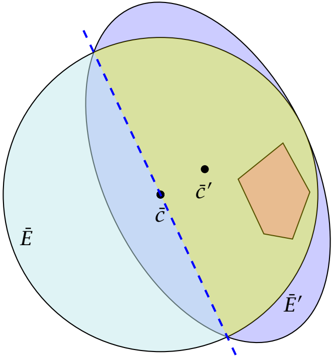

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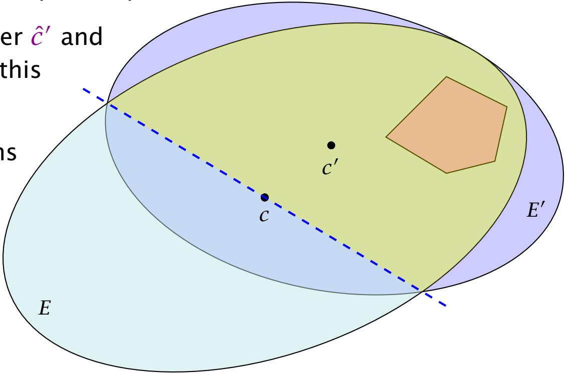
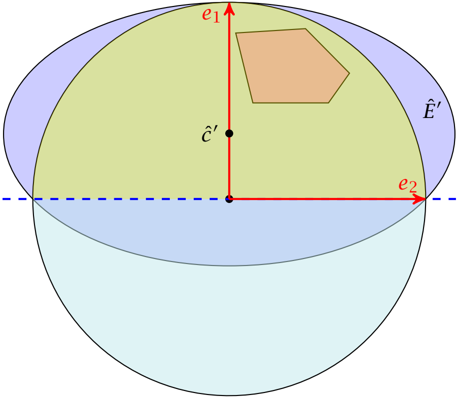
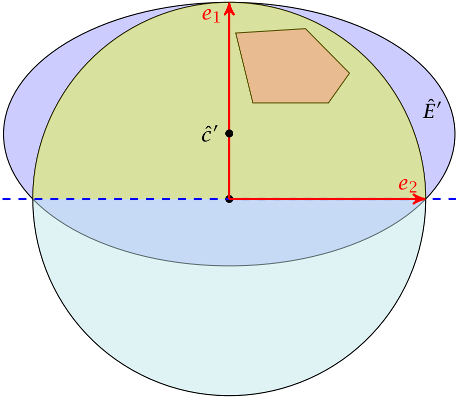


The Easy Case



- The new center lies on axis x_1. Hence, $\hat{c}' = te_1$ for $t > 0$.
- The vectors $e_1, e_2, \dots$ have to fulfill the ellipsoid constraint with equality. Hence $(e_i - \hat{c}')^T \hat{Q}'^{-1}(e_i - \hat{c}') = 1$.

The Easy Case



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The Easy Case

- The obtain the matrix $\hat{Q}'^{-1}$ for our ellipsoid $\hat{E}'$ note that $\hat{E}'$ is axis-parallel.
- Let a denote the radius along the x_1-axis and let b denote the (common) radius for the other axes.
- The matrix

$$\hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

maps the unit ball (via function $\hat{f}'(x) = \hat{L}'x$) to an axis-parallel ellipsoid with radius a in direction x_1 and b in all other directions.

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The Easy Case

- As $\hat{Q}' = \hat{L}'\hat{L}'^t$ the matrix $\hat{Q}'^{-1}$ is of the form

$$\hat{Q}'^{-1} = \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix}$$

The Easy Case

- $(e_1 - \hat{c}')^T \hat{Q}'^{-1} (e_1 - \hat{c}') = 1$ gives

$$\begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} 1-t \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $(1-t)^2 = a^2$.

The Easy Case

- For $i \neq 1$ the equation $(e_i - \hat{c}')^T \hat{Q}'^{-1} (e_i - \hat{c}') = 1$ gives

$$\begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{a^2} & 0 & \dots & 0 \\ 0 & \frac{1}{b^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b^2} \end{pmatrix} \cdot \begin{pmatrix} -t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1$$

- This gives $\frac{t^2}{a^2} + \frac{1}{b^2} = 1$, and hence

$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2}$$

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$$\frac{1}{b^2} = 1 - \frac{t^2}{a^2} = 1 - \frac{t^2}{(1-t)^2} = \frac{1-2t}{(1-t)^2}$$

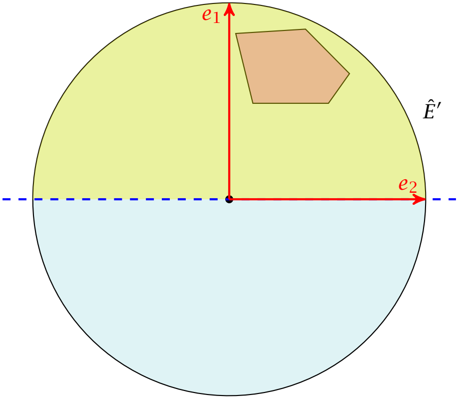
Summary

So far we have

$$a = 1 - t \quad \text{and} \quad b = \frac{1-t}{\sqrt{1-2t}}$$

The Easy Case

We still have many choices for t:



Choose t such that the volume of $\hat{E}'$ is minimal!!!

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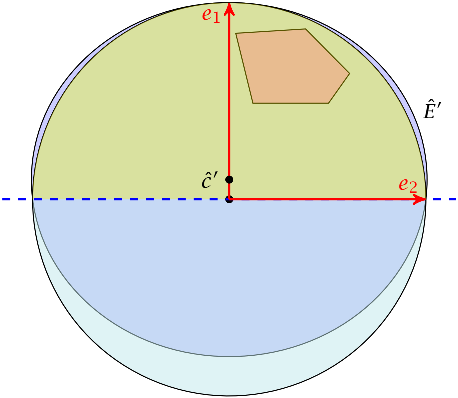

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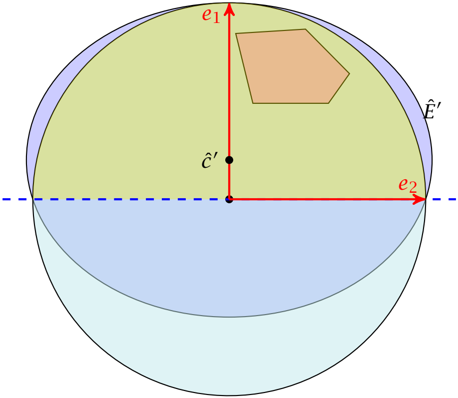

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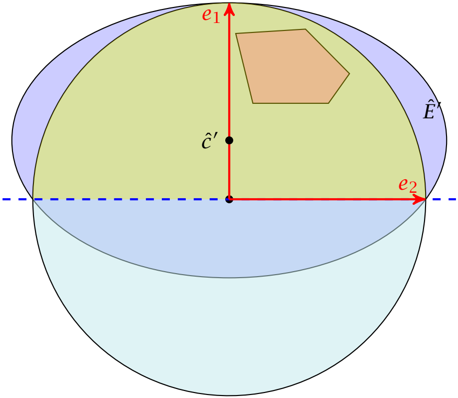

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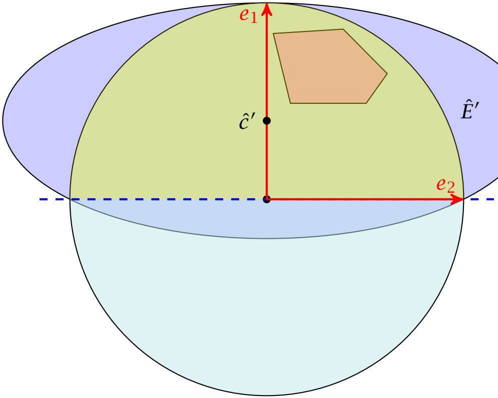

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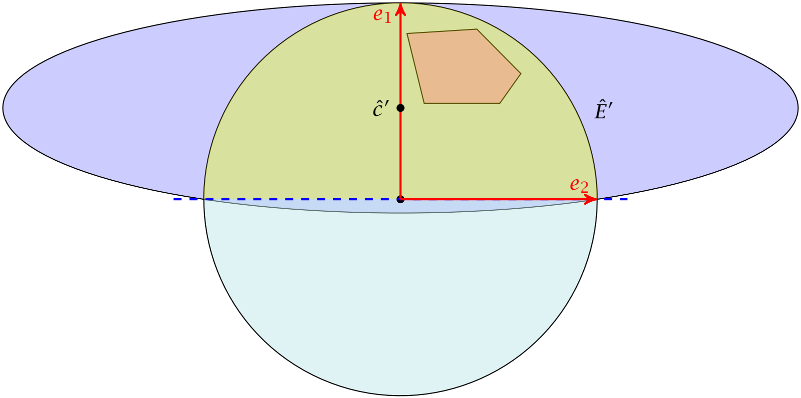

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The Easy Case

We want to choose t such that the volume of $\hat{E}'$ is minimal.

Lemma 51

Let L be an affine transformation and $K \subseteq \mathbb{R}^n$. Then

$$\mathrm{vol}(L(K)) = |\det(L)| \cdot \mathrm{vol}(K) \ .$$

The Easy Case

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Lemma 51

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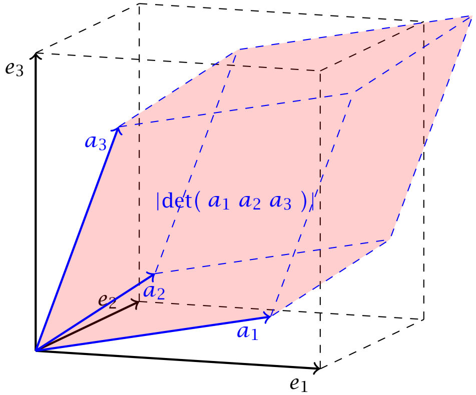
n-dimensional volume



The Easy Case

- We want to choose t such that the volume of $\hat{E}'$ is minimal.

$$\mathrm{vol}(\hat{E}') = \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \enspace ,$$

where $\hat{Q}' = \hat{L}'\hat{L}'^T$.

- We have

$$\hat{L}'^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & \dots & 0 \\ 0 & \frac{1}{b} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{b} \end{pmatrix} \text{ and } \hat{L}' = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & b & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b \end{pmatrix}$$

- Note that a and b in the above equations depend on t, by the previous equations.

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The Easy Case

$$\mathrm{vol}(\hat{E}')$$

The Easy Case

$$\mathrm{vol}(\hat{E}') = \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')|$$

The Easy Case

$$
\begin{aligned}
\mathrm{vol}(\hat{E}') &= \mathrm{vol}(B(0,1)) \cdot |\det(\hat{L}')| \\
&= \mathrm{vol}(B(0,1)) \cdot ab^{n-1}
\end{aligned}
$$

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\end{aligned}
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&= \mathrm{vol}(B(0,1)) \cdot \frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}
\end{aligned}
$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t}$$

The Easy Case

$$\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)$$

The Easy Case

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$$= \frac{1}{N^2}$$

$\boxed{N = \text{denominator}}$

The Easy Case

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$$= \frac{1}{N^2}\cdot\Bigg(\underbrace{(-1)\cdot n(1-t)^{n-1}}_{\text{derivative of numerator}}$$

The Easy Case

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$$-\underbrace{(n-1)(\sqrt{1-2t})^{n-2}}_{\text{outer derivative}}$$

The Easy Case

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$$= \frac{1}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1}$$

$$-(n-1)(\sqrt{1-2t})^{n-2}\cdot\underbrace{\frac{1}{2\sqrt{1-2t}}\cdot(-2)}_{\text{inner derivative}}$$

The Easy Case

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right) \\
&= \frac{1}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1} \\
&\qquad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\frac{1}{2\sqrt{1-2t}}\cdot(-2)\cdot\underbrace{(1-t)^n}_{\text{numerator}}\Bigg)
\end{aligned}
$$

The Easy Case

$$\begin{aligned}\frac{\mathrm{d}\,\mathrm{vol}(\hat{E}')}{\mathrm{d}\,t} &= \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{(1-t)^n}{(\sqrt{1-2t})^{n-1}}\right)\\ &= \frac{1}{N^2}\cdot\Bigg((-1)\cdot n(1-t)^{n-1}\cdot(\sqrt{1-2t})^{n-1}\\ &\quad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\frac{1}{2\sqrt{1-2t}}\cdot(-2)\cdot(1-t)^n\Bigg)\\ &= \frac{1}{N^2}\cdot(\sqrt{1-2t})^{n-3}\cdot(1-t)^{n-1}\end{aligned}$$

The Easy Case

$$
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&\quad -(n-1)(\sqrt{1-2t})^{n-2}\cdot\frac{1}{2\sqrt{1-2t}}\cdot(-2)\cdot(1-t)^n\Bigg) \\
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Let $\gamma_n = \frac{\mathrm{vol}(\hat{E}')}{\mathrm{vol}(B(0,1))} = ab^{n-1}$ be the ratio by which the volume changes:

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where we used $(1+x)^a \le e^{ax}$ for $x\in\mathbb{R}$ and $a>0$.

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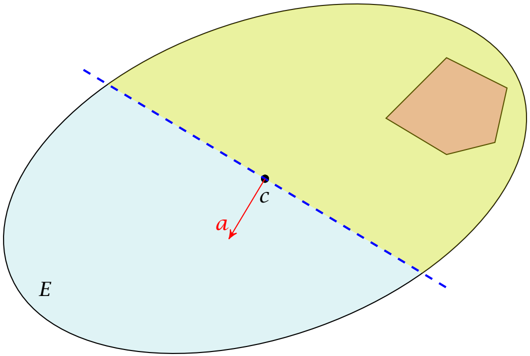
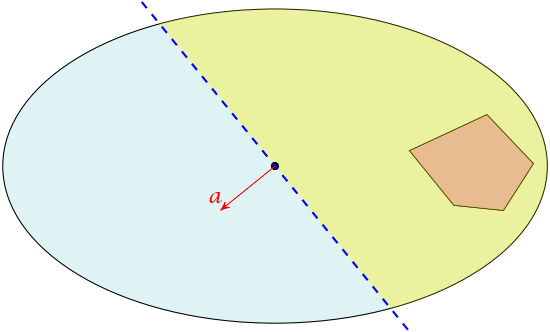
This gives $\gamma_n \le e^{-\frac{1}{2(n+1)}}$.

How to Compute the New Ellipsoid



How to Compute the New Ellipsoid

- Use f^{-1} (recall that $f = Lx + t$ is the affine transformation of the unit ball) to rotate/distort the ellipsoid (back) into the unit ball.



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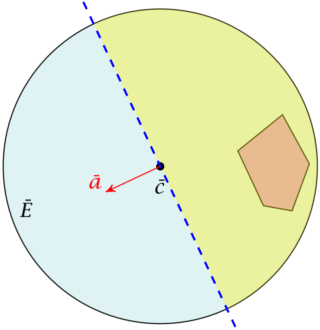

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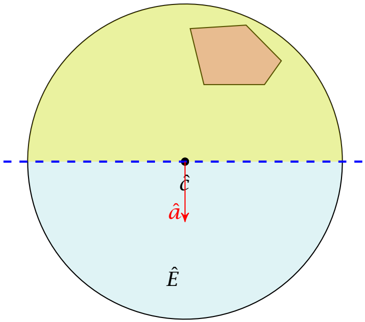

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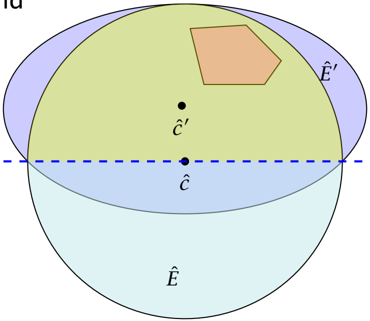

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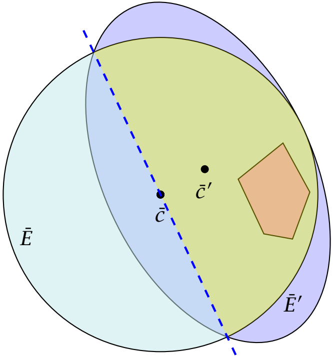

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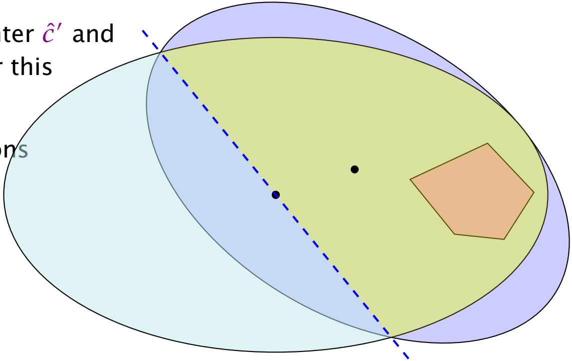

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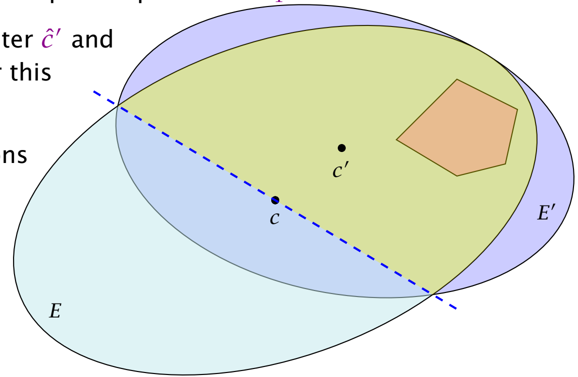

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Here it is important that mapping a set with affine function $f(x) = Lx + t$ changes the volume by factor $\det(L)$.

The Ellipsoid Algorithm

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This means $\bar{a} = L^T a$.

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After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

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Hence,

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$$R^{-1}\left(\frac{L^Ta}{\|L^Ta\|}\right) = -e_1 \quad \Rightarrow \quad -\frac{L^Ta}{\|L^Ta\|} = R \cdot e_1$$

Hence,

$$\bar{c}' = R \cdot \hat{c}' = R \cdot \frac{1}{n+1} e_1 = -\frac{1}{n+1}\frac{L^Ta}{\|L^Ta\|}$$

The Ellipsoid Algorithm

After rotating back (applying R^{-1}) the normal vector of the halfspace points in negative x_1-direction. Hence,

$$R^{-1}\Big(\frac{L^T a}{\|L^T a\|}\Big) = -e_1 \quad \Rightarrow \quad -\frac{L^T a}{\|L^T a\|} = R \cdot e_1$$

Hence,

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$$\begin{aligned} c' &= f(\bar{c}') = L \cdot \bar{c}' + c \\ &= -\frac{1}{n+1}L\frac{L^Ta}{\|L^Ta\|} + c \\ &= c - \frac{1}{n+1}\frac{Qa}{\sqrt{a^TQa}} \end{aligned}$$

For computing the matrix Q' of the new ellipsoid we assume in the following that $\hat{E}', \bar{E}'$ and E' refer to the ellispoids centered in the origin.

Recall that

$$\hat{Q}' = \begin{pmatrix} a^2 & 0 & \dots & 0 \\ 0 & b^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b^2 \end{pmatrix}$$

This gives

$$\hat{Q}' = \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1 e_1^T\Big)$$

because for $a = {}^{n}/_{n+1}$ and $b = {}^{n}/_{\sqrt{n^2-1}}$

$$b^2 - b^2\frac{2}{n+1} = \frac{n^2}{n^2-1} - \frac{2n^2}{(n-1)(n+1)^2}$$

$$= \frac{n^2(n+1) - 2n^2}{(n-1)(n+1)^2} = \frac{n^2(n-1)}{(n-1)(n+1)^2} = a^2$$

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This gives

$$\hat{Q}' = \frac{n^2}{n^2 - 1}\Big(I - \frac{2}{n+1} e_1 e_1^T\Big)$$

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9 The Ellipsoid Algorithm

$\bar{E}'$

9 The Ellipsoid Algorithm

$$\bar{E}' = R(\hat{E}')$$

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$$
\begin{aligned}
\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

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\bar{E}' &= R(\hat{E}') \\
&= \{R(x) \mid x^T \hat{Q}'^{-1} x \le 1\} \\
&= \{y \mid (R^{-1}y)^T \hat{Q}'^{-1} R^{-1} y \le 1\}
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&= \{y \mid y^T (\underbrace{R\hat{Q}'R^T}_{\bar{Q}'})^{-1} y \le 1\}
\end{aligned}
$$

9 The Ellipsoid Algorithm

Hence,

$$\bar{Q}'$$

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$$\begin{aligned}\bar{Q}' &= R\hat{Q}'R^T \\ &= R \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}e_1e_1^T\Big) \cdot R^T\end{aligned}$$

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&= \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^Taa^TL}{\|L^Ta\|^2}\Big)
\end{aligned}
$$

9 The Ellipsoid Algorithm

E'

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E' &= L(\bar{E}') \\
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Hence,

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Hence,

$$
\begin{aligned}
Q' &= L\bar{Q}'L^T \\
&= L \cdot \frac{n^2}{n^2-1}\Big(I - \frac{2}{n+1}\frac{L^T a a^T L}{a^T Q a}\Big) \cdot L^T \\
&= \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1}\frac{Q a a^T Q}{a^T Q a}\Big)
\end{aligned}
$$

Incomplete Algorithm

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$
output: point $x \in K$ or "K is empty"
$Q \leftarrow$???
repeat
  if $c \in K$ **then return** c
  else
    choose a violated hyperplane a
    $$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$
    $$Q \leftarrow \frac{n^2}{n^2-1}\Big(Q - \frac{2}{n+1} \frac{Qaa^TQ}{a^TQa}\Big)$$
  endif
until ???
return "K is empty"

Repeat: Size of basic solutions

Lemma 52

Let $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ be a bounded polyhedron. Let $\langle a_{\max}\rangle$ be the maximum encoding length of an entry in A, b. Then every entry x_j in a basic solution fulfills $|x_j| = \frac{D_j}{D}$ with $D_j, D \le 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

In the following we use $\delta := 2^{2n\langle a_{\max}\rangle + 2n\log_2 n}$.

Note that here we have $P = \{x \mid Ax \le b\}$. The previous lemmas we had about the size of feasible solutions were slightly different as they were for different polytopes.

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Repeat: Size of basic solutions

Proof:

Let $\bar{A} = \begin{bmatrix} A & -A & I_m \\ -A & A & \end{bmatrix}$, $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$, be the matrix and right-hand vector after transforming the system to standard form.

The determinant of the matrices $\bar{A}_B$ and $\bar{M}_j$ (matrix obt. when replacing the j-th column of $\bar{A}_B$ by $\bar{b}$) can become at most

$$\begin{aligned} \det(\bar{A}_B), \det(\bar{M}_j) &\leq \|\vec{\ell}_{\max}\|^{2n} \\ &\leq (\sqrt{2n} \cdot 2^{\langle a_{\max} \rangle})^{2n} \leq 2^{2n\langle a_{\max}\rangle + 2n\log_2 n} \end{aligned},$$

where $\vec{\ell}_{\max}$ is the longest column-vector that can be obtained after deleting all but $2n$ rows and columns from $\bar{A}$.

This holds because columns from I_m selected when going from $\bar{A}$ to $\bar{A}_B$ do not increase the determinant. Only the at most $2n$ columns from matrices A and $-A$ that $\bar{A}$ consists of contribute.

How do we find the first ellipsoid?

For feasibility checking we can assume that the polytop P is bounded; it is sufficient to consider basic solutions.

Every entry x_i in a basic solution fulfills $|x_i| \leq \delta$.

Hence, P is contained in the cube $-\delta \leq x_i \leq \delta$.

A vector in this cube has at most distance $R := \sqrt{n}\delta$ from the origin.

Starting with the ball $E_0 := B(0, R)$ ensures that P is completely contained in the initial ellipsoid. This ellipsoid has volume at most $R^n B(0,1) \leq (n\delta)^n B(0,1)$.

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When can we terminate?

Let $P := \{x \mid Ax \le b\}$ with $A \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be a bounded polytop. Let $\langle a_{\max} \rangle$ be the encoding length of the largest entry in A or b.

Consider the following polyhedron

$$P_\lambda := \left\{ x \mid Ax \le b + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\} ,$$

where $\lambda = \delta^2 + 1$.

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Lemma 53

P_λ is feasible if and only if P is feasible.

$\Longleftarrow$: obvious!

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$\Longrightarrow$:

Consider the polyhedrons

$$\bar{P} = \left\{x \mid \begin{bmatrix} A & -A & \\ -A & A & I_m \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix}; x \geq 0 \right\}$$

and

$$\bar{P}_\lambda = \left\{x \mid \begin{bmatrix} A & -A & \\ -A & A & I_m \end{bmatrix} x = \begin{pmatrix} b \\ -b \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}; x \geq 0 \right\} .$$

P is feasible if and only if $\bar{P}$ is feasible, and P_λ feasible if and only if $\bar{P}_\lambda$ feasible.

$\bar{P}_\lambda$ is bounded since P_λ and P are bounded.

$\Longrightarrow$:

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Let $\bar{A} = \begin{bmatrix} A & -A & I_m \\ -A & A & \end{bmatrix}$, and $\bar{b} = \begin{pmatrix} b \\ -b \end{pmatrix}$.

$\bar{P}_\lambda$ feasible implies that there is a basic feasible solution represented by

$$x_B = \bar{A}_B^{-1}\bar{b} + \frac{1}{\lambda}\bar{A}_B^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

(The other x-values are zero)

The only reason that this basic feasible solution is not feasible for $\bar{P}$ is that one of the basic variables becomes negative.

Hence, there exists i with

$$(\bar{A}_B^{-1}\bar{b})_i < 0 \leq (\bar{A}_B^{-1}\bar{b})_i + \frac{1}{\lambda}(\bar{A}_B^{-1}\vec{1})_i$$

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and

$$(\bar{A}_B^{-1}\vec{1})_i \le \det(\bar{M}_j) \ ,$$

where $\bar{M}_j$ is obtained by replacing the j-th column of $\bar{A}_B$ by $\vec{1}$.

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Lemma 54

If P_λ *is feasible then it contains a ball of radius* $r := 1/\delta^3$. *This has a volume of at least* $r^n \mathrm{vol}(B(0,1)) = \frac{1}{\delta^{3n}} \mathrm{vol}(B(0,1))$.

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Hence, $x + \vec{\ell}$ is feasible for P_λ which proves the lemma.

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&= \mathcal{O}(\mathrm{poly}(n, \langle a_{\max}\rangle))
\end{aligned}$$

Algorithm 1 ellipsoid-algorithm

input: point $c \in \mathbb{R}^n$, convex set $K \subseteq \mathbb{R}^n$, radii R and r
    with $K \subseteq B(c, R)$, and $B(x, r) \subseteq K$ for some x
output: point $x \in K$ or "K is empty"
$Q \leftarrow \mathrm{diag}(R^2, \ldots, R^2)$ // i.e., $L = \mathrm{diag}(R, \ldots, R)$
repeat
    if $c \in K$ **then return** c
    else
        choose a violated hyperplane a

$$c \leftarrow c - \frac{1}{n+1} \frac{Qa}{\sqrt{a^T Q a}}$$

$$Q \leftarrow \frac{n^2}{n^2 - 1}\left(Q - \frac{2}{n+1} \frac{Qaa^T Q}{a^T Q a}\right)$$

    endif
until $\det(Q) \le r^{2n}$ // i.e., $\det(L) \le r^n$
return "K is empty"

Separation Oracle:

Let $K \subseteq \mathbb{R}^n$ be a convex set. A separation oracle for K is an algorithm A that gets as input a point $x \in \mathbb{R}^n$ and either

- certifies that $x \in K$,
- or finds a hyperplane separating x from K.

We will usually assume that A is a polynomial-time algorithm.

In order to find a point in K we need

- a guarantee that a ball of radius r is contained in K,
- an initial ball $B(c, R)$ with radius R that contains K,
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The Ellipsoid algorithm requires $\mathcal{O}(\text{poly}(n) \cdot \log(R/r))$ iterations. Each iteration is polytime for a polynomial-time Separation oracle.

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- inequalities $Ax \le b$; $m \times n$ matrix A with rows a_i^T
- $P = \{x \mid Ax \le b\}$; $P^\circ := \{x \mid Ax < b\}$
- interior point algorithm: $x \in P^\circ$ throughout the algorithm
- for $x \in P^\circ$ define

$$s_i(x) := b_i - a_i^T x$$

  as the slack of the i-th constraint

logarithmic barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \log(s_i(x))$$

Penalty for point x; points close to the boundary have a very large penalty.

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picture of barrier function

Gradient and Hessian

Taylor approximation:

$$\phi(x+\epsilon) \approx \phi(x) + \nabla\phi(x)^T\epsilon + \frac{1}{2}\epsilon^T\nabla^2\phi(x)\epsilon$$

Gradient:

$$\nabla\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} \cdot a_i = A^T d_x$$

where $d_x^T = (1/s_1(x), \ldots, 1/s_m(x))$. ($d_x$ vector of inverse slacks)

Hessian:

$$H_x := \nabla^2\phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)^2} a_i a_i^T = A^T D_x^2 A$$

with $D_x = \mathrm{diag}(d_x)$.

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$$
\begin{aligned}
\frac{\partial \phi(x)}{\partial x_i} &= \frac{\partial}{\partial x_i}\left(-\sum_r w_r \ln(s_r(x))\right) \\
&= -\sum_r w_r \frac{\partial}{\partial x_i}\left(\ln(s_r(x))\right) \\
&= -\sum_r w_r \frac{1}{s_r(x)} \frac{\partial}{\partial x_i}\left(s_r(x)\right) \\
&= -\sum_r w_r \frac{1}{s_r(x)} \frac{\partial}{\partial x_i}\left(b_r - a_r^T x\right) \\
&= \sum_r w_r \frac{1}{s_r(x)} \frac{\partial}{\partial x_i}\left(a_r^T x\right) \\
&= \sum_r w_r \frac{1}{s_r(x)} A_{ri}
\end{aligned}
$$

The i-th entry of the gradient vector is $\sum_r w_r/s_r(x) \cdot A_{ri}$. This gives that the gradient is

$$
\nabla \phi(x) = \sum_r w_r/s_r(x) a_r = A^T D_x W \vec{1}
$$

$$\frac{\partial}{\partial x_j}\left(\sum_r \frac{w_r}{s_r(x)} A_{ri}\right) = \sum_r w_r A_{ri}\left(-\frac{1}{s_r(x)^2}\right) \cdot \frac{\partial}{\partial x_j}\left(s_r(x)\right)$$
$$= \sum_r w_r A_{ri} \frac{1}{s_r(x)^2} A_{rj}$$

Note that $\sum_r A_{ri} A_{rj} = (A^T A)_{ij}$. Adding the additional factors $w_r / s_r(x)^2$ can be done with a diagonal matrix.

Hence the Hessian is

$$H_x = A^T D W D A$$

H_x is positive semi-definite for $x \in P^\circ$

$$u^T H_x u = u^T A^T D_x^2 A u = \|D_x A u\|_2^2 \geq 0$$

This gives that $\phi(x)$ is convex.

If $\mathrm{rank}(A) = n$, H_x is positive definite for $x \in P^\circ$

$$u^T H_x u = \|D_x A u\|_2^2 > 0 \text{ for } u \neq 0$$

This gives that $\phi(x)$ is strictly convex.

$\|u\|_{H_x} := \sqrt{u^T H_x u}$ is a (semi-)norm; the unit ball w.r.t. this norm is an ellipsoid.

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Dilkin Ellipsoid

$$E_x = \{y \mid (y-x)^T H_x (y-x) \le 1\} = \{y \mid \|y-x\|_{H_x} \le 1\}$$

Points in E_x are feasible!!!

$$(y-x)^T H_x (y-x) = (y-x)^T A^T S_x^{-2} A (y-x)$$

$$= \sum_i \frac{(a_i^T(y-x))^2}{s_i(x)^2}$$

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In order to become infeasible when going from x to y one of the terms in the sum would need to be larger than 1.

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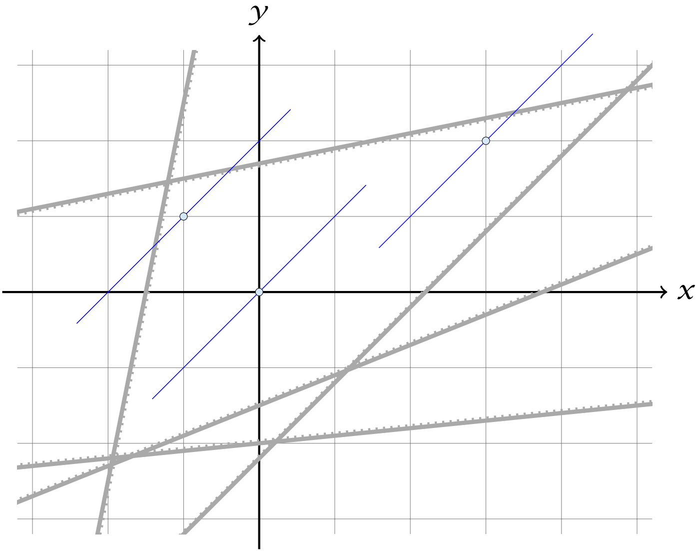

Analytic Center

$$x_{\mathrm{ac}} := \arg\min_{x \in P^\circ} \phi(x)$$

- x_{ac} is solution to

$$\nabla \phi(x) = \sum_{i=1}^{m} \frac{1}{s_i(x)} a_i = 0$$

- depends on the description of the polytope
- x_{ac} exists and is unique iff P° is nonempty and bounded

Central Path

In the following we assume that the LP and its dual are strictly feasible and that $\operatorname{rank}(A) = n$.

Central Path:
Set of points $\{x^*(t) \mid t > 0\}$ with

$$x^*(t) = \operatorname{argmin}_x \{tc^T x + \phi(x)\}$$

- $t = 0$: analytic center
- $t = \infty$: optimum solution

$x^*(t)$ exists and is unique for all $t \geq 0$.

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primal-dual pair:

$$\begin{array}{ll} \text{min} & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

$$\begin{array}{ll} \text{max} & -b^T z \\ \text{s.t.} & A^T z + c = 0 \\ & z \geq 0 \end{array}$$

we assume primal and dual problems are strictly feasible; $\mathrm{rank}(A) = n$.

Point $x^*(t)$ on central path is solution to $tc + \nabla\phi(x) = 0$ (force field interpretation).

This means

$$tc + \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x^*(t)} = 0$$

or

$$c + \sum_{i=1}^{m} z_i^*(t) a_i = 0 \quad \text{with} \quad z_i^*(t) = \frac{1}{t(b_i - a_i^T x^*(t))}$$

- $z_i^*(t)$ is strictly dual feasible
- duality gap between $x := x^*(t)$ and $z := z^*(t)$ is

$$c^T x + b^T z = (b - Ax)^T z = \frac{m}{t}$$

- if this gap is less than $1/\Omega(2^L)$ we can snap to an optimum point

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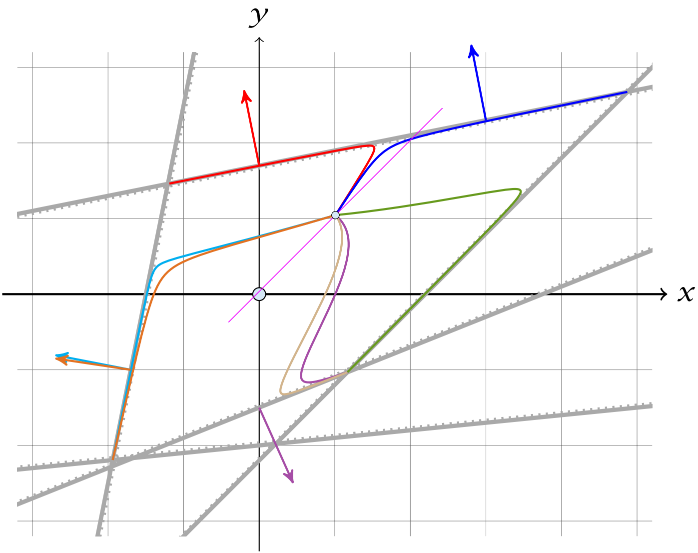

y
x

Path-following Methods

Algorithm 1 PathFollowing

1: start at analytic center
2: **while** solution not good enough **do**
3: make step to improve objective function
4: recenter to return to central path

Questions/Remarks

- how do we get to analytic center?
- when is solution “good enough”?
- (usually) improvement step tries to stay feasible, how?
- recentering step should
  - be fast
  - not undo (too much of) improvement

Centering Problem

$$\text{minimize } f_t(x) = tc^T x + \phi(x)$$

minimizing this gives point $x^*(t)$ on central path

Newton Step at $x \in P^\circ$

$$\begin{aligned} \Delta x_{\text{nt}} &= -H^{-1}\nabla f_t(x) \\ &= -H^{-1}(tc + \nabla \phi(x)) \\ &= -(A^T D_x^2 A)^{-1}(tc + A^T d_x) \end{aligned}$$

Newton Iteration:

$$x := x + \Delta x_{\text{nt}}$$

Measuring Progress of Newton Step

Newton decrement:

$$\begin{aligned}\lambda_t(x) &= \|D_x A \Delta x_{\mathsf{nt}}\| \\ &= \|\Delta x_{\mathsf{nt}}\|_{H_x}\end{aligned}$$

Square of Newton decrement is linear estimate of reduction if we do a Newton step:

$$-\lambda_t(x)^2 = \nabla f_t(x)^T \Delta x_{\mathsf{nt}}$$

- $\lambda_t(x) = 0$ iff $x = x^*(t)$
- $\lambda_t(x)$ is measure of proximity of x to $x^*(t)$

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Convergence of Newtons Method

Theorem 55
If $\lambda_t(x) < 1$ *then*

- $x_+ := x + \Delta x_{nt} \in P^\circ$ *(new point feasible)*
- $\lambda_t(x_+) \le \lambda_t(x)^2$

This means we have quadratic convergence. Very fast.

feasibility:

- $\lambda_t(x) = \|\Delta x_{\mathsf{nt}}\| < 1$; hence x_+ lies in the Dilkin ellipsoid around x.

bound on $\lambda_t(x^+)$:

we use $D := D_x = \operatorname{diag}(d_x)$ and $D_+ := D_{x^+} = \operatorname{diag}(d_{x^+})$

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&= \|(I - D_+^{-1} D) D A \Delta x_{\mathrm{nt}}\|^2
\end{aligned}
$$

To see the last equality

$$
\begin{aligned}
\|a\|^2 + \|a+b\|^2 &= a^T a + (a^T + b^T)(a+b) \\
&= (a^T + b^T)a + a^T(a+b) + b^T b = \|b\|^2
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if $a^T(a+b) = 0$.

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&\quad = \Delta x_{\mathsf{nt}}^{+T} \Big(H_+ \Delta x_{\mathsf{nt}}^+ - H \Delta x_{\mathsf{nt}} + A^T D_+ \vec{1} - A^T D \vec{1} \Big) \\
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\begin{aligned}
a^T(a+b) \\
&= \Delta x_{\mathsf{nt}}^{+T} A^T D_+ \left(D_+ A \Delta x_{\mathsf{nt}}^+ + (I - D_+^{-1}D) DA\Delta x_{\mathsf{nt}} \right) \\
&= \Delta x_{\mathsf{nt}}^{+T} \left(A^T D_+^2 A \Delta x_{\mathsf{nt}}^+ - A^T D^2 A \Delta x_{\mathsf{nt}} + A^T D_+ DA\Delta x_{\mathsf{nt}} \right) \\
&= \Delta x_{\mathsf{nt}}^{+T} \left(H_+ \Delta x_{\mathsf{nt}}^+ - H \Delta x_{\mathsf{nt}} + A^T D_+ \vec{1} - A^T D \vec{1} \right) \\
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bound on $\lambda_t(x^+)$:
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\lambda_t(x^+)^2 &= \|D_+ A \Delta x_{\mathrm{nt}}^+\|^2 \\
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&\le \|(I - D_+^{-1} D) \vec{1}\|^4 \\
&= \|D A \Delta x_{\mathrm{nt}}\|^4 \\
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Short step barrier method

simplifying assumptions:

- a first central point $x^*(t_0)$ is given
- $x^*(t)$ is computed exactly in each iteration

ϵ is approximation we are aiming for

start at $t = t_0$, repeat until $m/t \leq \epsilon$

- compute $x^*(\mu t)$ using Newton starting from $x^*(t)$
- $t := \mu t$

where $\mu = 1 + 1/(2\sqrt{m})$

gradient of f_{t^+} at ($x = x^*(t)$)

$$\begin{aligned}\nabla f_{t^+}(x) &= \nabla f_t(x) + (\mu - 1)tc \\ &= -(\mu - 1)A^T D_x \vec{1}\end{aligned}$$

This holds because $0 = \nabla f_t(x) = tc + A^T D_x \vec{1}$.

The Newton decrement is

$$\begin{aligned}\lambda_{t^+}(x)^2 &= (\nabla f_{t^+}(x))^T H^{-1} \nabla f_{t^+}(x) \\ &= (\mu - 1)^2 \vec{1}^T B(B^T B)^{-1} B^T \vec{1} \qquad B = D_x A \\ &\le (\mu - 1)^2 m \\ &= 1/4\end{aligned}$$

This means we are in the range of quadratic convergence!!!

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Number of Iterations

the number of Newton iterations per outer iteration is very small; in practise only 1 or 2

Number of outer iterations:
We need $t_k = \mu^k t_0 \geq m/\epsilon$. This holds when

$$k \geq \frac{\log(m/(\epsilon t_0))}{\log(\mu)}$$

We get a bound of

$$\mathcal{O}\left(\sqrt{m}\log\frac{m}{\epsilon t_0}\right)$$

We show how to get a starting point with $t_0 = 1/2^L$. Together with $\epsilon \approx 2^L$ we get $\mathcal{O}(L\sqrt{m})$ iterations.

How to start...

a damped Newton iteration goes in the direction of Δx_{nt} but only so far as to not leave the polytope;

Lemma 56 (without proof)

A damped Newton iteration starting at x_0 reaches a point with $\lambda_t(x) \le \delta$ after

$$\frac{f_t(x_0) - \min_y f_t(y)}{0.09} + \mathcal{O}(\log\log(1/\delta))$$

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This will allow us to quickly reach a point on the central path ($t \approx 2^L$) when starting very close to it (e.g. at the analytic center).

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How to get close to analytic center?

Let $P = \{Ax \leq b\}$ be our (feasible) polyhedron, and x_0 a feasible point.

We change $b \to b + \frac{1}{\lambda} \cdot \vec{1}$, where $L = L(A, b)$ (encoding length) and $\lambda = 2^{2L}$. Recall that a basis is feasible in the old LP iff it is feasible in the new LP.

After, re-normalizing A and b (for integrality) we have that the point x_0 is at distance at least 1 from every constraint.

The determinant of the matrix A_B for a basis B went up by a factor of 2^{2nL}.

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Start at x_0.

Choose $c' := -\nabla\phi(x)$.

$x_0 = x^*(1)$ is point on central path for c' and $t = 1$.

You can travel the central path in both directions. Go towards 0 until $t \approx 1/2^{nL}$. This requires $\sqrt{m}nL$ outer iterations.

Let $x_{c'}$ denote this point.

Let x_c denote the point that minimizes

$$t \cdot c^T x + \phi(x)$$

(i.e., same value for t but different c, hence, different central path).

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&\le t \cdot c^T x_c + \phi(x_c) + 2t 2^{\langle c_{\max} \rangle} 2^{nL}
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Choosing $t = 1/2^{\Omega(nL)}$ gives that the last term becomes very small. Hence, using damped Newton we can move to a point on the new central path (for c) quickly.

In total for this analysis we require $\mathcal{O}(\sqrt{m}nL)$ outer iterations for the whole algorithm.

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