Simulations between PRAMs

Theorem 1
We can simulate a p-processor priority CRCW PRAM on a p-processor EREW PRAM with slowdown $\mathcal{O}(\log p)$.

Simulations between PRAMs

Theorem 2
We can simulate a p-processor priority CRCW PRAM on a $p \log p$-processor common CRCW PRAM with slowdown $\mathcal{O}(1)$.

Simulations between PRAMs

Theorem 3
We can simulate a p-processor priority CRCW PRAM on a p-processor common CRCW PRAM with slowdown $\mathcal{O}(\frac{\log p}{\log \log p})$.

Simulations between PRAMs

Theorem 4
We can simulate a p-processor priority CRCW PRAM on a p-processor arbitrary CRCW PRAM with slowdown $\mathcal{O}(\log \log p)$.

Lower Bounds for the CREW PRAM

Ideal PRAM:

- every processor has unbounded local memory
- in each step a processor reads a global variable
- then it does some (unbounded) computation on its local memory
- then it writes a global variable

Lower Bounds for the CREW PRAM

Definition 5

An input index i affects a memory location M at time t on some input I if the content of M at time t differs between inputs I and $I(i)$ (i-th bit flipped).

$L(M,t,I) = \{i \mid i \text{ affects } M \text{ at time } t \text{ on input } I\}$

Lower Bounds for the CREW PRAM

Definition 6

An input index i affects a processor P at time t on some input I if the state of P at time t differs between inputs I and $I(i)$ (i-th bit flipped).

$K(P,t,I) = \{i \mid i \text{ affects } P \text{ at time } t \text{ on input } I\}$

Lower Bounds for the CREW PRAM

Lemma 7

If $i \in K(P,t,I)$ with $t > 1$ then either

- $i \in K(P,t-1,I)$, *or*
- P *reads a global memory location* M *on input* I *at time* t, *and* $i \in L(M,t-1,I)$.

Lower Bounds for the CREW PRAM

Lemma 8

If $i \in L(M,t,I)$ with $t > 1$ then either

- *A processor writes into M at time t on input I and $i \in K(P,t,I)$, or*
- *No processor writes into M at time t on input I and*
  - *either $i \in L(M,t-1,I)$*
  - *or a processor P writes into M at time t on input $I(i)$.*

Let $k_0 = 0, \ell_0 = 1$ and define

$$k_{t+1} = k_t + \ell_t \text{ and } \ell_{t+1} = 3k_t + 4\ell_t$$

Lemma 9

$|K(P,t,I)| \leq k_t$ *and* $|L(M,t,I)| \leq \ell_t$ *for any* $t \geq 0$

base case ($t = 0$):

- No index can influence the local memory/state of a processor before the first step (hence $|K(P,0,I)| = k_0 = 0$).
- Initially every index in the input affects exactly one memory location. Hence $|L(M,0,I)| = 1 = \ell_0$.

induction step ($t \to t+1$):

$K(P,t+1,I) \subseteq K(P,t,I) \cup L(M,t,I)$, where M is the location read by P in step $t+1$.

Hence,

$$\begin{aligned}|K(P,t+1,I)| &\leq |K(P,t,I)| + |L(M,t,I)| \\ &\leq k_t + \ell_t\end{aligned}$$

induction step ($t \to t+1$):

For the bound on $|L(M, t+1, I)|$ we have two cases.

Case 1:
A processor P writes into location M at time $t+1$ on input I.

Then,

$$\begin{aligned}|L(M, t+1, I)| &\leq |K(P, t+1, I)| \\ &\leq k_t + \ell_t \\ &\leq 3k_t + 4\ell_t = \ell_{t+1}\end{aligned}$$

Case 2:
No processor P writes into location M at time $t+1$ on input I.

An index i affects M at time $t+1$ iff i affects M at time t or some processor P writes into M at $t+1$ on $I(i)$.

$L(M, t+1, I) \subseteq L(M, t, I) \cup Y(M, t+1, I)$

$Y(M, t+1, I)$ is the set of indices u_j that cause some processor P_{w_j} to write into M at time $t+1$ on input I.

$Y(M, t+1, I)$ is the set of indices u_j that cause some processor P_{w_j} to write into M at time $t+1$ on input I.

Fact:
For all pairs u_s, u_t with $P_{w_s} \neq P_{w_t}$ either
$u_s \in K(P_{w_t}, t+1, I(u_t))$ or $u_t \in K(P_{w_s}, t+1, I(u_s))$.

Otherwise, P_{w_t} and P_{w_s} would both write into M at the same time on input $I(u_s)(u_t)$.

Let $U = \{u_1, \dots, u_r\}$ denote all indices that cause some processor to write into M.

Let $V = \{(I(u_1), P_{w_1}), \dots\}$.

We set up a bipartite graph between U and V, such that $(u_i, (I(u_j), P_{w_j})) \in E$ if u_i affects P_{w_j} at time $t+1$ on input $I(u_j)$.

Each vertex $(I(u_j), P_{w_j})$ has degree at most k_{t+1} as this is an upper bound on indices that can influence a processor P_{w_j}.

Hence, $|E| \leq r \cdot k_{t+1}$.

For an index u_j there can be at most k_{t+1} indices u_i with $P_{w_i} = P_{w_j}$.

Hence, there must be at least $\frac{1}{2}r(r - k_{t+1})$ pairs u_i, u_j with $P_{w_i} \neq P_{w_j}$.

Each pair introduces at least one edge.

Hence,

$$|E| \geq \frac{1}{2}r(r - k_{t+1})$$

This gives $r \leq 3k_{t+1} \leq 3k_t + 3\ell_t$

Recall that $L(M, t+1, i) \subseteq L(M, t, i) \cup Y(M, t+1, I)$

$|L(M, t+1, i)| \leq 3k_t + 4\ell_t$

$$\begin{pmatrix} k_{t+1} \\ \ell_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} k_t \\ \ell_t \end{pmatrix} \qquad \begin{pmatrix} k_0 \\ \ell_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \frac{1}{2}(5 + \sqrt{21}) \text{ and } \lambda_2 = \frac{1}{2}(5 - \sqrt{21})$$

Eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ -(1 - \lambda_1) \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ -(1 - \lambda_2) \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ \frac{3}{2} + \frac{1}{2}\sqrt{21} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ \frac{3}{2} - \frac{1}{2}\sqrt{21} \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ \frac{3}{2} + \frac{1}{2}\sqrt{21} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ \frac{3}{2} - \frac{1}{2}\sqrt{21} \end{pmatrix}$$

$$\begin{pmatrix} k_0 \\ \ell_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{21}}(v_1 - v_2)$$

$$\begin{pmatrix} k_t \\ \ell_t \end{pmatrix} = \frac{1}{\sqrt{21}}\left(\lambda_1^t v_1 - \lambda_2^t v_2\right)$$

Solving the recurrence gives

$$k_t = \frac{\lambda_1^t}{\sqrt{21}} - \frac{\lambda_2^t}{\sqrt{21}}$$

$$\ell_t = \frac{3+\sqrt{21}}{2\sqrt{21}}\lambda_1^t + \frac{-3+\sqrt{21}}{2\sqrt{21}}\lambda_2^t$$

with $\lambda_1 = \frac{1}{2}(5+\sqrt{21})$ and $\lambda_2 = \frac{1}{2}(5-\sqrt{21})$.

Theorem 10

The following problems require logarithmic time on a CREW PRAM.

- *Sorting a sequence of $x_1, \dots, x_n$ with $x_i \in \{0,1\}$*
- *Computing the maximum of n inputs*
- *Computing the sum $x_1 + \cdots + x_n$ with $x_i \in \{0,1\}$*

A Lower Bound for the EREW PRAM

Definition 11 (Zero Counting Problem)

Given a monotone binary sequence $x_1, x_2, \dots, x_n$ determine the index i such that $x_i = 0$ and $x_{i+1} = 1$.

We show that this problem requires $\Omega(\log n - \log p)$ steps on a p-processor EREW PRAM.

Let I_i be the input with i zeros folled by $n - i$ ones.

Index i affects processor P at time t if the state in step t is differs between I_{i-1} and I_i.

Index i affects location M at time t if the content of M after step t differs between inputs I_{i-1} and I_i.

Lemma 12

If $i \in K(P,t)$ then either

- *$i \in K(P,t-1)$, or*
- *P reads some location M on input I_i (and, hence, also on I_{i-1}) at step t and $i \in L(M,t-1)$*

Lemma 13

If $i \in L(M,t)$ then either

- *$i \in L(M,t-1)$, or*
- *Some processor P writes M at step t on input I_i and $i \in K(P,t)$.*
- *Some processor P writes M at step t on input I_{i-1} and $i \in K(P,t)$.*

Define

$$C(t) = \sum_P |K(P,t)| + \sum_M \max\{0, |L(M,t)| - 1\}$$

$C(T) \geq n$, $C(0) = 0$

Claim:
$C(t) \leq 6C(t-1) + 3|P|$

This gives $C(T) \leq \frac{6^T-1}{5} 3|P|$ and hence $T = \Omega(\log n - \log |P|)$.

For an index i to newly appear in $L(M,t)$ some processor must write into M on either input I_i or I_{i-1}.

Hence, any index in $K(P,t)$ can at most generate two new indices in $L(M,t)$.

This means that the number of new indices in any set $L(M,t)$ (over all M) is at most

$$2\sum_P |K(P,t)|$$

Hence,

$$\sum_M |L(M,t)| \le \sum_M |L(M,t-1)| + 2\sum_P |K(P,t)|$$

We can assume wlog. that $L(M,t-1) \subseteq L(M,t)$. Then

$$\sum_M \max\{0, |L(M,t)| - 1\} \le \sum_M \max\{0, |L(M,t-1)| - 1\} + 2\sum_P |K(P,t)|$$

For an index i to newly appear in $K(P,t)$, P must read a memory location M with $i \in L(M,t)$ on input I_i (and also on input I_{i-1}).

Since we are in the EREW model at most one processor can do so in every step.

Let $J(i,t)$ be memory locations read in step t on input I_i, and let $J_t = \bigcup_i J(i,t)$.

$$\sum_P |K(P,t)| \le \sum_P |K(P,t-1)| + \sum_{M \in J_t} |L(M,t-1)|$$

Over all inputs I_i a processor can read at most $|K(P,t-1)| + 1$ different memory locations (why?).

Hence,

$$\begin{aligned}\sum_P |K(P,t)| &\le \sum_P |K(P,t-1)| + \sum_{M \in J_t} |L(M,t-1)| \\ &\le \sum_P |K(P,t-1)| + \sum_{M \in J_t} (|L(M,t-1)| - 1) + J_t \\ &\le 2\sum_P |K(P,t-1)| + \sum_{M \in J_t} (|L(M,t-1)| - 1) + |P| \\ &\le 2\sum_P |K(P,t-1)| + \sum_M \max\{0, |L(M,t-1)| - 1\} + |P|\end{aligned}$$

Recall

$$\sum_M \max\{0, |L(M,t)| - 1\} \le \sum_M \max\{0, |L(M,t-1)| - 1\} + 2\sum_P |K(P,t)|$$

This gives

$$\begin{aligned}&\sum_P K(P,t) + \sum_M \max\{0, |L(M,t)| - 1\} \\ &\quad \le 4\sum_M \max\{0, |L(M,t-1)| - 1\} + 6\sum_P |K(P,t-1)| + 3|P|\end{aligned}$$

Hence,

$$C(t) \le 6C(t-1) + 3|P|$$

Lower Bounds for CRCW PRAMS

Theorem 14

Let $f : \{0,1\}^n \to \{0,1\}$ be an arbitrary Boolean function. f can be computed in $\mathcal{O}(1)$ time on a common CRCW PRAM with $\le n2^n$ processors.

Can we obtain non-constant lower bounds if we restrict the number of processors to be polynomial?

Boolean Circuits

- nodes are either AND, OR, or NOT gates or are special INPUT/OUTPUT nodes
- AND and OR gates have unbounded fan-in (indegree) and ounbounded fan-out (outdegree)
- NOT gates have unbounded fan-out
- INPUT nodes have indegree zero; OUTPUT nodes have outdegree zero
- size is the number of edges
- depth is the longest path from an input to an output

Theorem 15

Let $f : \{0,1\}^n \to \{0,1\}^m$ be a function with n inputs and $m \le n$ outputs, and circuit C computes f with depth $D(n)$ and size $S(n)$. Then f can be computed by a common CRCW PRAM in $\mathcal{O}(D(n))$ time using $S(n)$ processors.

Given a family $\{C_n\}$ of circuits we may not be able to compute the corresponding family of functions on a CRCW PRAM.

Definition 16

A family $\{C_n\}$ of circuits is logspace uniform if there exists a deterministic Turing machine M s.t

- M runs in logarithmic space.
- For all n, M outputs C_n on input 1^n.