Prefix Sum

input: $x[1] \dots x[n]$
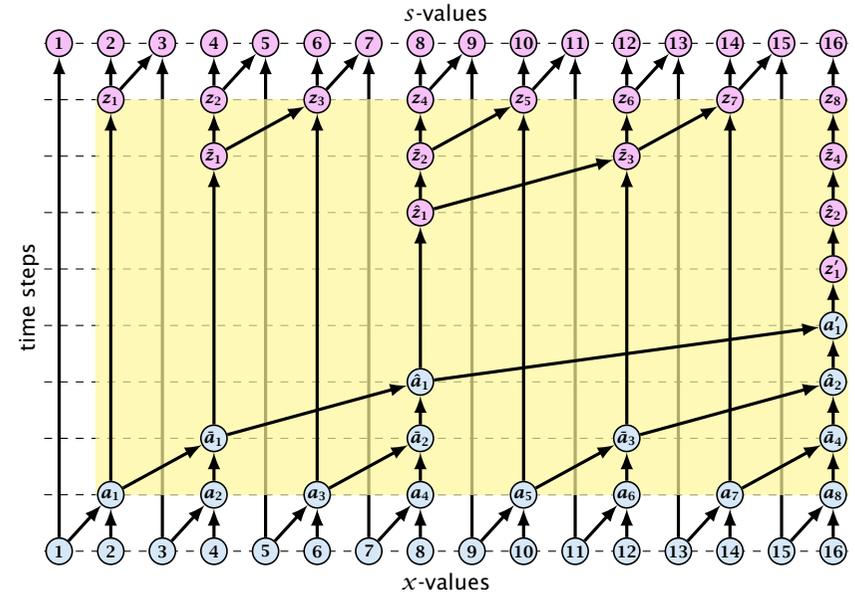output: $s[1] \dots s[n]$ with $s[i] = \sum_{j=1}^{i} x[i]$ (w.r.t. operator $*$)

Algorithm 6 PrefixSum$(n, x[1] \dots x[n])$

1: // compute prefixsums; $n = 2^k$
2: **if** $n = 1$ **then** $s[1] \leftarrow x[1]$; **return**
3: **for** $1 \le i \le n/2$ **pardo**
4:      $a[i] \leftarrow x[2i-1] * x[2i]$
5: $z[1], \dots, z[n/2] \leftarrow$ PrefixSum$(n/2, a[1] \dots a[n/2])$
6: **for** $1 \le i \le n$ **pardo**
7:      i even   : $s[i] \leftarrow z[i/2]$
8:      $i = 1$: $s[1] = x[1]$
9:      i odd   : $s[i] \leftarrow z[(i-1)/2] * x[i]$

Prefix Sum

Prefix Sum

The algorithm uses work $\mathcal{O}(n)$ and time $\mathcal{O}(\log n)$ for solving Prefix Sum on an EREW-PRAM with n processors.
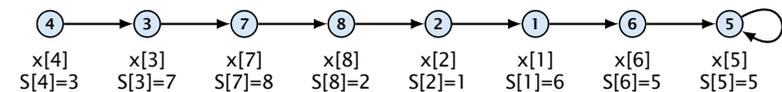
It is clearly work-optimal.

Theorem 1
On a CREW PRAM a Prefix Sum requires running time $\Omega(\log n)$ regardless of the number of processors.

Parallel Prefix

Input: a linked list given by successor pointers; a value $x[i]$ for every list element; an operator $*$;

Output: for every list position ℓ the sum (w.r.t. $*$) of elements after ℓ in the list (including ℓ)

Parallel Prefix

> **Algorithm 7** ParallelPrefix
> 1: **for** $1 \le i \le n$ **pardo**
> 2: $P[i] \leftarrow S[i]$
> 3:   **while** $S[i] \neq S[S[i]]$ **do**
> 4: $x[i] \leftarrow x[i] * x[S[i]]$
> 5: $S[i] \leftarrow S[S[i]]$
> 6:   **if** $P[i] \neq i$ **then** $S[i] \leftarrow x[S(i)]$

The algorithm runs in time $\mathcal{O}(\log n)$.

It has work requirement $\mathcal{O}(n \log n)$. non-optimal

This technique is also known as pointer jumping

4.3 Divide & Conquer — Merging

Given two sorted sequences $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, compute the sorted squence $C = (c_1, \ldots, c_n)$.

Definition 2
Let $X = (x_1, \ldots, x_t)$ be a sequence. The rank rank$(y : X)$ of y in X is
$$\text{rank}(y : X) = |\{x \in X \mid x \le y\}|$$

For a sequence $Y = (y_1, \ldots, y_s)$ we define
rank$(Y : X) := (r_1, \ldots, r_s)$ with $r_i = \text{rank}(y_i : X)$.

4.3 Divide & Conquer — Merging

Given two sorted sequences $A = (a_1 \ldots a_n)$ and $B = (b_1 \ldots b_n)$, compute the sorted squence $C = (c_1 \ldots c_n)$.

Observation:
We can assume wlog. that elements in A and B are different.

Then for $c_i \in C$ we have $i = \text{rank}(c_i : A \cup B)$.

This means we just need to determine rank$(x : A \cup B)$ for all elements!

Observe, that $\text{rank}(x : A \cup B) = \text{rank}(x : A) + \text{rank}(x : B)$.

Clearly, for $x \in A$ we already know $\text{rank}(x : A)$, and for $x \in B$ we know $\text{rank}(x : B)$.

4.3 Divide & Conquer — Merging

Compute $\text{rank}(x : A)$ for all $x \in B$ and $\text{rank}(x : B)$ for all $x \in A$.
can be done in $\mathcal{O}(\log n)$ time with $2n$ processors by binary search

Lemma 3
On a CREW PRAM, Merging can be done in $\mathcal{O}(\log n)$ time and $\mathcal{O}(n \log n)$ work.

not optimal

4.3 Divide & Conquer — Merging

$A = (a_1, \ldots, a_n)$; $B = (b_1, \ldots, b_n)$;
$\log n$ integral; $k := n/\log n$ integral;

Algorithm 8 GenerateSubproblems

1: $j_0 \leftarrow 0$
2: $j_k \leftarrow n$
3: **for** $1 \leq i \leq k-1$ **pardo**
4:       $j_i \leftarrow \text{rank}(b_{i\log n} : A)$
5: **for** $0 \leq i \leq k-1$ **pardo**
6:       $B_i \leftarrow (b_{i\log n+1}, \ldots, b_{(i+1)\log n})$
7:       $A_i \leftarrow (a_{j_i+1}, \ldots, a_{j_{i+1}})$

If C_i is the merging of A_i and B_i then the sequence $C_0 \ldots C_{k-1}$ is a sorted sequence.

4.3 Divide & Conquer — Merging

We can generate the subproblems in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$.

Note that in a sub-problem B_i has length $\log n$.

If we run the algorithm again for every subproblem, (where A_i takes the role of B) we can in time $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n)$ generate subproblems where A_j and B_j have both length at most $\log n$.

Such a subproblem can be solved by a single processor in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(|A_i| + |B_i|)$.

Parallelizing the last step gives total work $\mathcal{O}(n)$ and time $\mathcal{O}(\log n)$.

the resulting algorithm is work optimal

4.4 Maximum Computation

Lemma 4
On a CRCW PRAM the maximum of n numbers can be computed in time $\mathcal{O}(1)$ with n^2 processors.

proof on board...

4.4 Maximum Computation

Lemma 5
On a CRCW PRAM the maximum of n numbers can be computed in time $\mathcal{O}(\log\log n)$ with n processors and work $\mathcal{O}(n\log\log n)$.

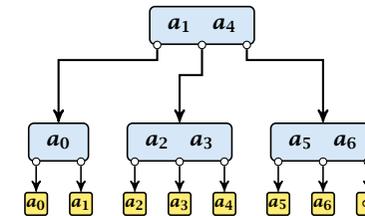proof on board...

4.4 Maximum Computation

Lemma 6

On a CRCW PRAM the maximum of n numbers can be computed in time $\mathcal{O}(\log\log n)$ with n processors and work $\mathcal{O}(n)$.

proof on board...

4.5 Inserting into a $(2,3)$-tree

Given a $(2,3)$-tree with n elements, and a sequence $x_0 < x_1 < x_2 < \cdots < x_k$ of elements. We want to insert elements $x_1, \ldots, x_k$ into the tree ($k \ll n$).
**time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k \log n)$**

4.5 Inserting into a $(2,3)$-tree

1. determine for every x_i the leaf element before which it has to be inserted
   time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k \log n)$; CREW PRAM

   all x_i's that have to be inserted before the same element form a chain

2. determine the largest/smallest/middle element of every chain
   time: $\mathcal{O}(1)$; work: $\mathcal{O}(k)$;

3. insert the middle element of every chain
   compute new chains
   time: $\mathcal{O}(\log n)$; work: $\mathcal{O}(k_i \log n)$; k_i= #inserted elements
   (computing new chains is constant time)

4. repeat Step 3 for logarithmically many rounds
   time: $\mathcal{O}(\log n \log k)$; work: $\mathcal{O}(k \log n)$;

Step 3
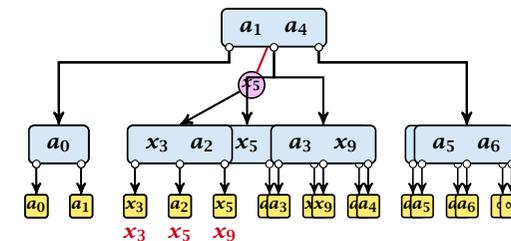
▶ each internal node is split into at most two parts
▶ each split operation promotes at most one element
▶ hence, on every level we want to insert at most one element per successor pointer
▶ we can use the same routine for every level

4.5 Inserting into a $(2, 3)$-tree

- ▶ Step 3, works in phases; one phase for every level of the tree
- ▶ Step 4, works in rounds; in each round a different set of elements is inserted

Observation
We can start with phase i of round r as long as phase i of round $r - 1$ and (of course), phase $i - 1$ of round r has finished.

This is called Pipelining. Using this technique we can perform all rounds in Step 4 in just $\mathcal{O}(\log k + \log n)$ many parallel steps.
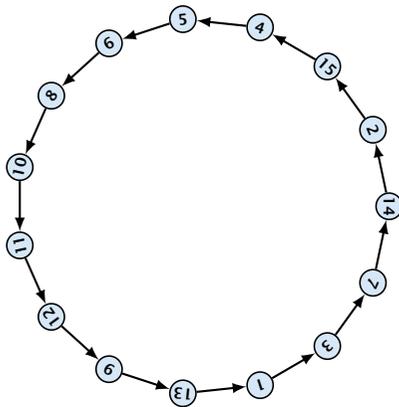
4.6 Symmetry Breaking

The following algorithm colors an n-node cycle with $\lceil \log n \rceil$ colors.

Algorithm 9 BasicColoring
1: **for** $1 \le i \le n$ **pardo**
2:     $\mathrm{col}(i) \leftarrow i$
3:     $k_i \leftarrow$ smallest bitpos where $\mathrm{col}(i)$ and $\mathrm{col}(S(i))$ differ
4:     $\mathrm{col}'(i) \leftarrow 2k + \mathrm{col}(i)_k$

4.6 Symmetry Breaking

v	col	k	col$'$
1	0001	1	2
3	0011	2	4
7	0111	0	1
14	1110	2	5
2	0010	0	0
15	1111	0	1
4	0100	0	0
5	0101	0	1
6	0110	1	3
8	1000	1	2
10	1010	0	0
11	1011	0	1
12	1100	0	0
9	1001	2	4
13	1101	2	5

4.6 Symmetry Breaking

Applying the algorithm to a coloring with bit-length t generates a coloring with largest color at most

$$2(t - 1) + 1$$

and bit-length at most

$$\lceil \log_2(2(t - 1) + 1) \rceil \le \lceil \log_2(t - 1) \rceil + 1 \le \lceil \log_2(t) \rceil + 1$$

Applying the algorithm repeatedly generates a constant number of colors after $\log^* n$ operations.

> Note that the first inequality holds because $2(t - 1) - 1$ is odd.

4.6 Symmetry Breaking

As long as the bit-length $t \geq 4$ the bit-length decreases.

Applying the algorithm with bit-length 3 gives a coloring with colors in the range $0, \ldots, 5 = 2t - 1$.

We can improve to a 3-coloring by successively re-coloring nodes from a color-class:

Algorithm 10 ReColor

1: **for** $\ell \leftarrow 5$ **to** 3
2:      **for** $1 \leq i \leq n$ **pardo**
3:          **if** $\mathrm{col}(i) = \ell$ **then**
4:              $\mathrm{col}(i) \leftarrow \min\{\{0, 1, 2\} \setminus \{\mathrm{col}(P[i]), \mathrm{col}(S[i])\}\}$

This requires time $\mathcal{O}(1)$ and work $\mathcal{O}(n)$.

4.6 Symmetry Breaking

Lemma 7
We can color vertices in a ring with three colors in $\mathcal{O}(\log^* n)$ time and with $\mathcal{O}(n \log^* n)$ work.

not work optimal

4.6 Symmetry Breaking

Lemma 8
Given n integers in the range $0, \ldots, \mathcal{O}(\log n)$, there is an algorithm that sorts these numbers in $\mathcal{O}(\log n)$ time using a linear number of operations.

Proof: Exercise!

4.6 Symmetry Breaking

Algorithm 11 OptColor

1: **for** $1 \leq i \leq n$ **pardo**
2:      $\mathrm{col}(i) \leftarrow i$
3: apply BasicColoring once
4: sort vertices by colors
5: **for** $\ell = 2\lceil \log n \rceil$ **to** 3 **do**
6:      **for** all vertices i of color ℓ **pardo**
7:          $\mathrm{col}(i) \leftarrow \min\{\{0, 1, 2\} \setminus \{\mathrm{col}(P[i]), \mathrm{col}(S[i])\}\}$

We can perform Lines 6 and 7 in time $\mathcal{O}(n_\ell)$ only because we sorted before. In general a statement like "**for** constraint **pardo**" should only contain a contraint on the id's of the processors but not something complicated (like the color) which has to be checked and, hence, induces work. Because of the sorting we can transform this complicated constraint into a constraint on just the processor id's.

Lemma 9

A ring can be colored with 3 colors in time $\mathcal{O}(\log n)$ and with work $\mathcal{O}(n)$.

work optimal but not too fast