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An extension of binary search with p processors gives that one can find the rank of an element in

$$\log_{p+1}(n) = \frac{\log n}{\log(p+1)}$$

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Given two sorted sequences $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$, compute the sorted squence $C = (c_1, \ldots, c_n)$.

Definition 1

Let $X = (x_1, \ldots, x_t)$ be a sequence. The rank $\mathrm{rank}(y : X)$ of y in X is

$$\mathrm{rank}(y : X) = |\{x \in X \mid x \leq y\}|$$

For a sequence $Y = (y_1, \ldots, y_s)$ we define $\mathrm{rank}(Y : X) := (r_1, \ldots, r_s)$ with $r_i = \mathrm{rank}(y_i : X)$.

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We have already seen a merging-algorithm that runs in time $\mathcal{O}(\log n)$ and work $\mathcal{O}(n)$.

Using the fast search algorithm we can improve this to a running time of $\mathcal{O}(\log\log n)$ and work $\mathcal{O}(n \log\log n)$.

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Merging

Input: $A = a_1, \ldots, a_n$; $B = b_1, \ldots, b_m$; $m \leq n$

1. if $m < 4$ then rank elements of B, using the parallel search algorithm with p processors. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(n)$.
2. Concurrently rank elements $b_{\sqrt{m}}, b_{2\sqrt{m}}, \ldots, b_m$ in A using the parallel search algorithm with $p = \sqrt{n}$. Time: $\mathcal{O}(1)$. Work: $\mathcal{O}(\sqrt{m} \cdot \sqrt{n}) = \mathcal{O}(n)$

   $j(i) := \mathrm{rank}(b_{i\sqrt{m}} : A)$
3. Let $B_i = (b_{i\sqrt{m}+1}, \ldots, b_{(i+1)\sqrt{m}-1})$; and $A_i = (a_{j(i)+1}, \ldots, a_{j(i+1)})$.

   Recursively compute $\mathrm{rank}(B_i : A_i)$.
4. Let k be index not a multiple of $\sqrt{m}$. $i = \lceil \frac{k}{\sqrt{m}} \rceil$. Then $\mathrm{rank}(b_k : A) = j(i) + \mathrm{rank}(b_k : A_i)$.

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The algorithm can be made work-optimal by standard techniques.

proof on board...

Mergesort

Lemma 2

A straightforward parallelization of Mergesort can be implemented in time $\mathcal{O}(\log n \log\log n)$ and with work $\mathcal{O}(n \log n)$.

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Mergesort

Let $L[v]$ denote the (sorted) sublist of elements stored at the leaf nodes rooted at v.

We can view Mergesort as computing $L[v]$ for a complete binary tree where the leaf nodes correspond to nodes in the given array.

Since the merge-operations on one level of the complete binary tree can be performed in parallel we obtain time $\mathcal{O}(h \log\log n)$ and work $\mathcal{O}(hn)$, where $h = \mathcal{O}(\log n)$ is the height of the tree.

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Pipelined Mergesort

We again compute $L[v]$ for every node in the complete binary tree.

After round s, $L_s[v]$ is an **approximation** of $L[v]$ that will be improved in future rounds.

For $s \geq 3\,\mathrm{height}(v)$, $L_s[v] = L[v]$.

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In every round, a node v sends $\text{sample}(L_s[v])$ (an approximation of its current list) upwards, and receives approximations of the lists of its children.

It then computes a new approximation of its list.

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Algorithm 11 ColeSort()

1: initialize $L_0[v] = A_v$ for leaf nodes; $L_0[v] = \emptyset$ otw.
2: **for** $s \leftarrow 1$ **to** $3 \cdot \text{height}(T)$ **do**
3: **for** all active nodes v **do**
4: // u and w children of v
5: $L'_s[u] \leftarrow \text{sample}(L_{s-1}[u])$
6: $L'_s[w] \leftarrow \text{sample}(L_{s-1}[w])$
7: $L_s[v] \leftarrow \text{merge}(L'_s[u], L'_s[u])$

$$\text{sample}(L_s[v]) = \begin{cases} \text{sample}_4(L_s[v]) & s \leq 3\,\text{height}(v) \\ \text{sample}_2(L_s[v]) & s = 3\,\text{height}(v) + 1 \\ \text{sample}_1(L_s[v]) & s = 3\,\text{height}(v) + 2 \end{cases}$$

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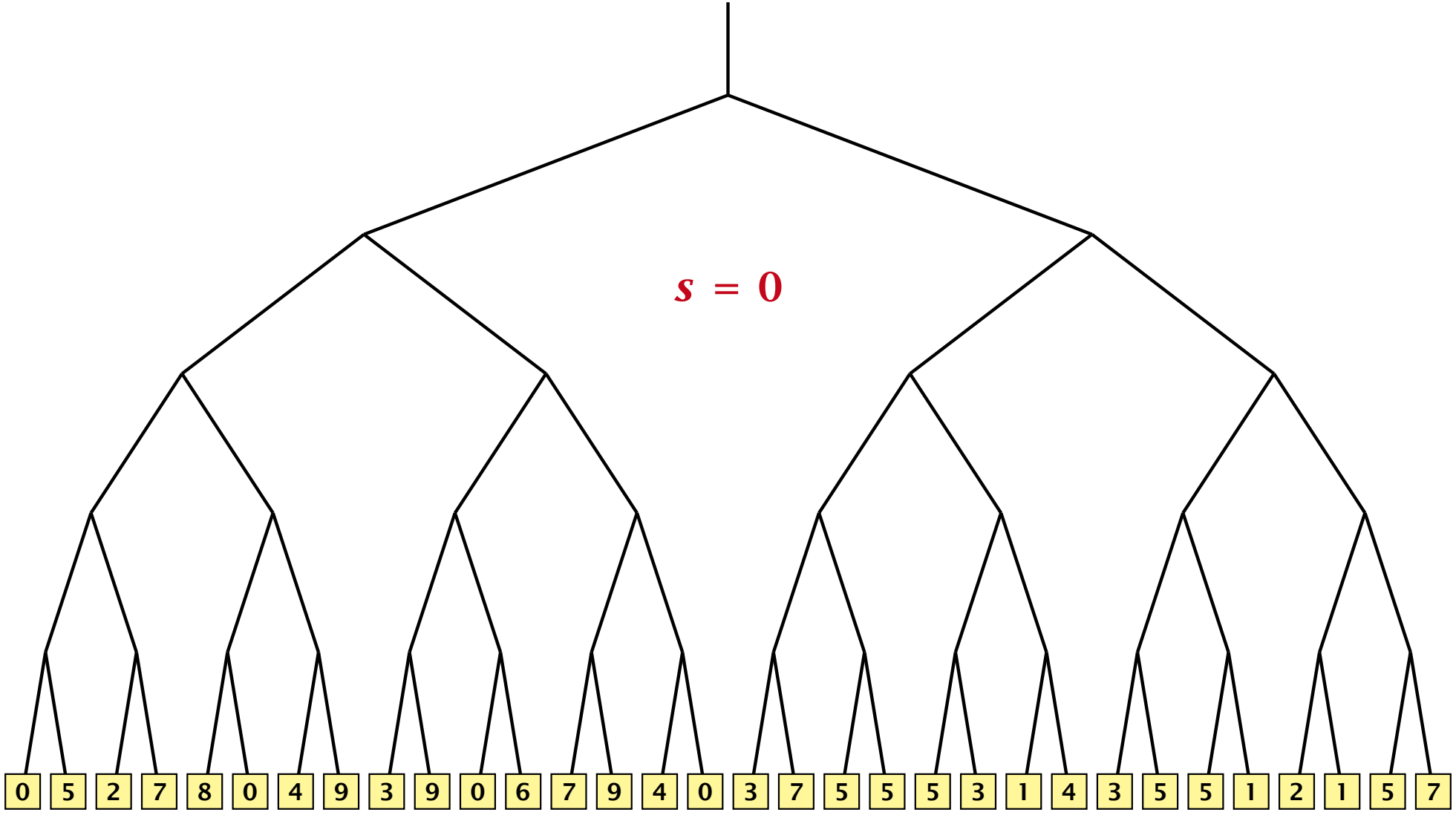
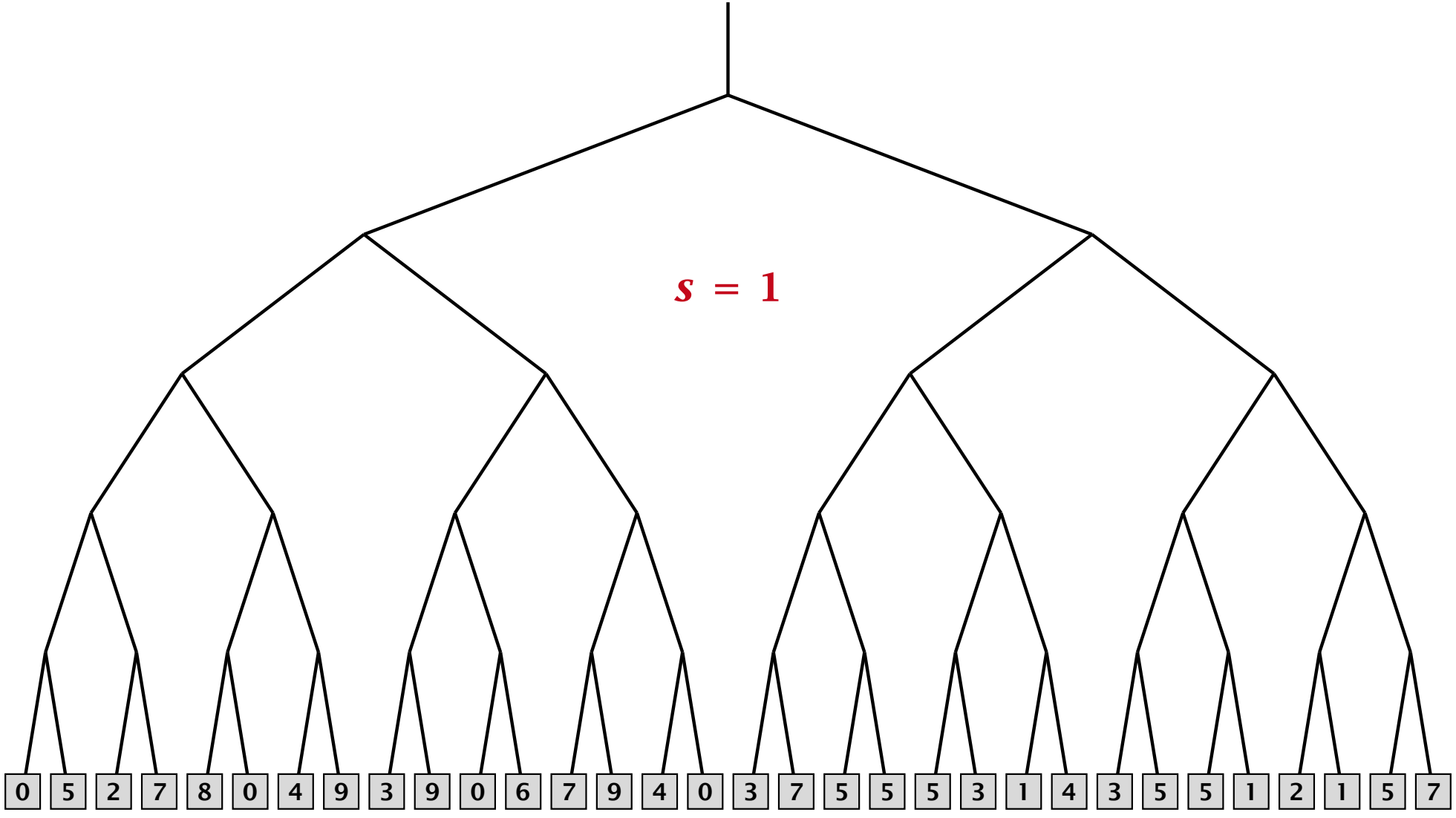
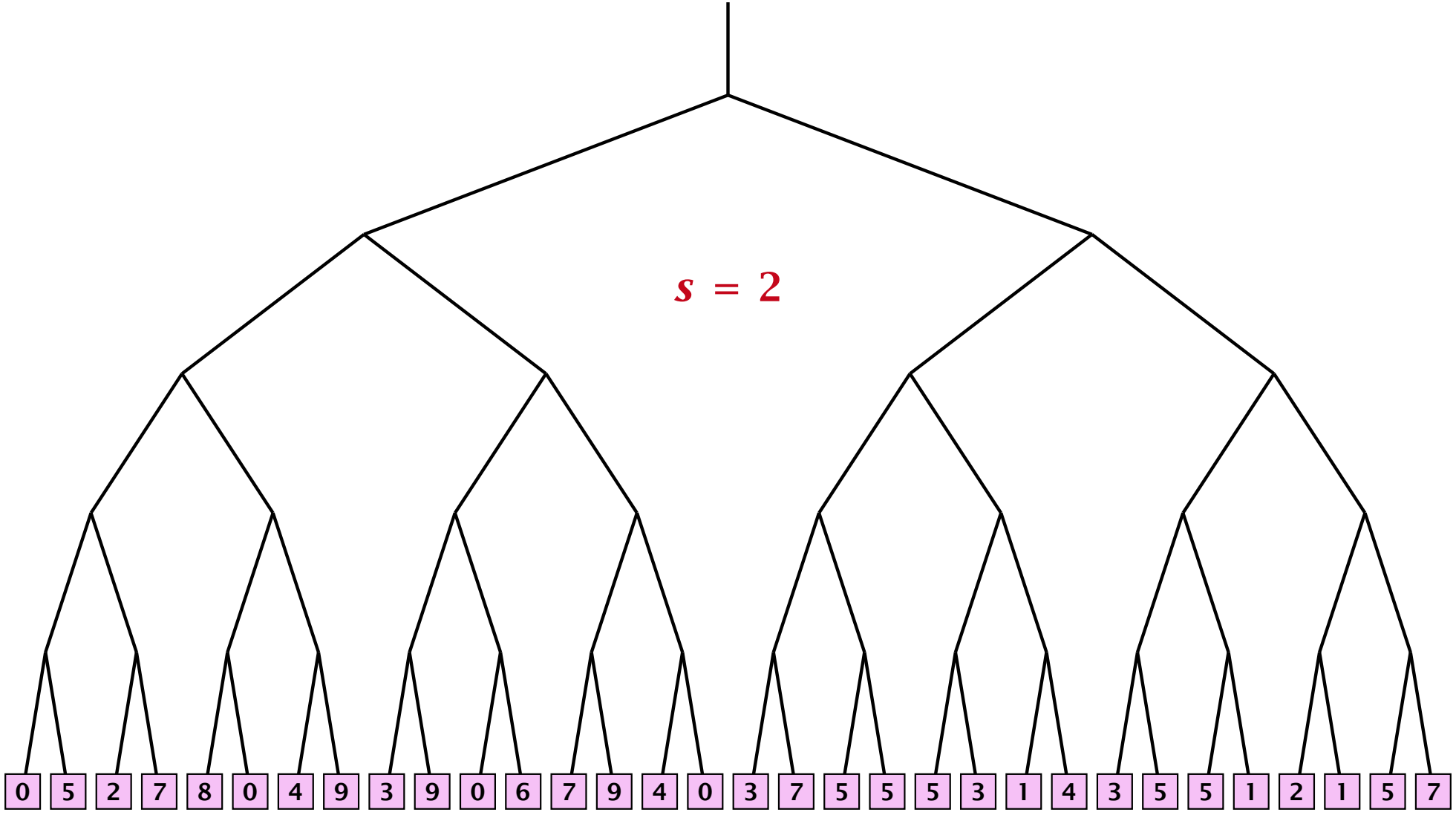
Colesort



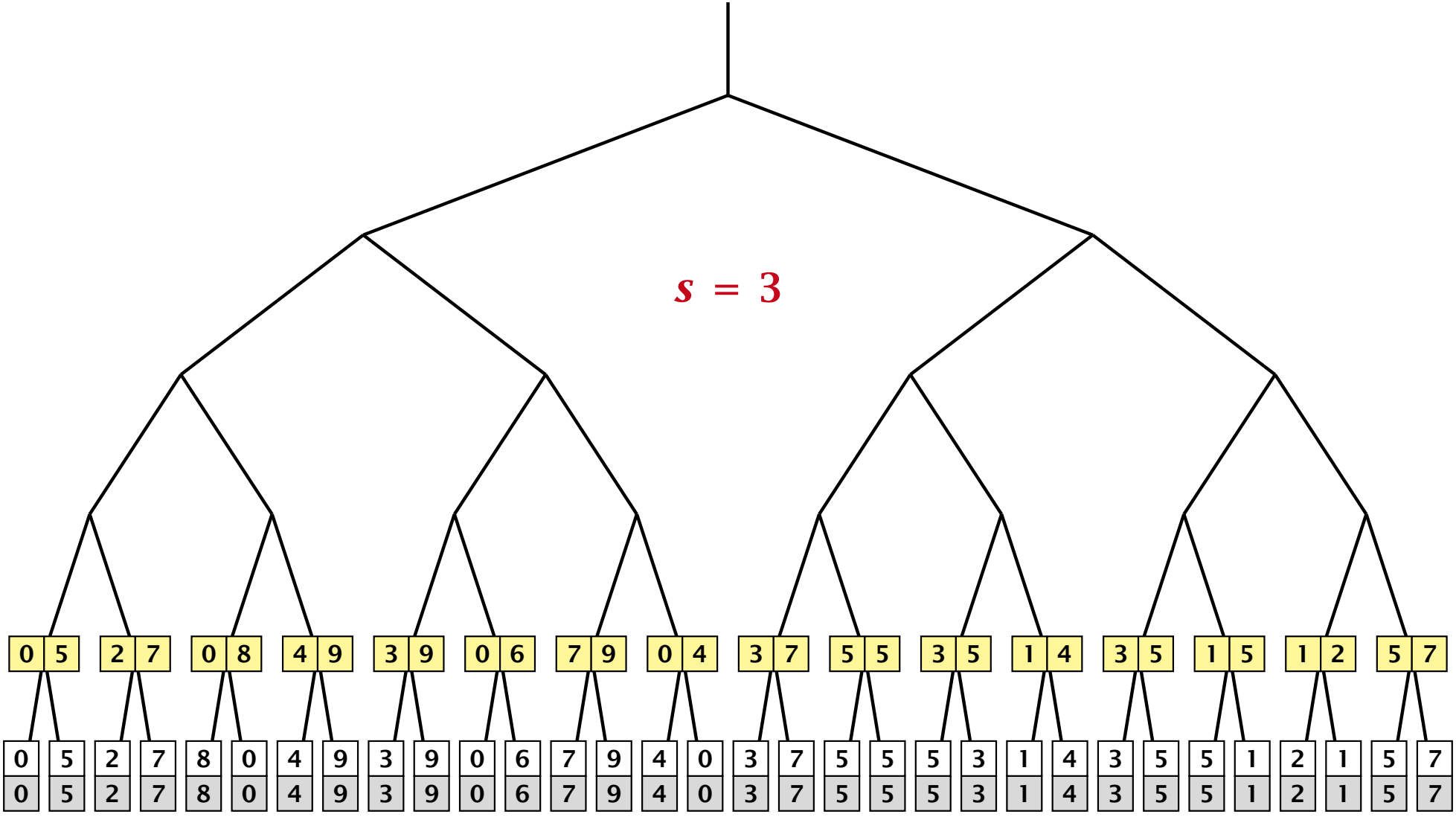
Colesort



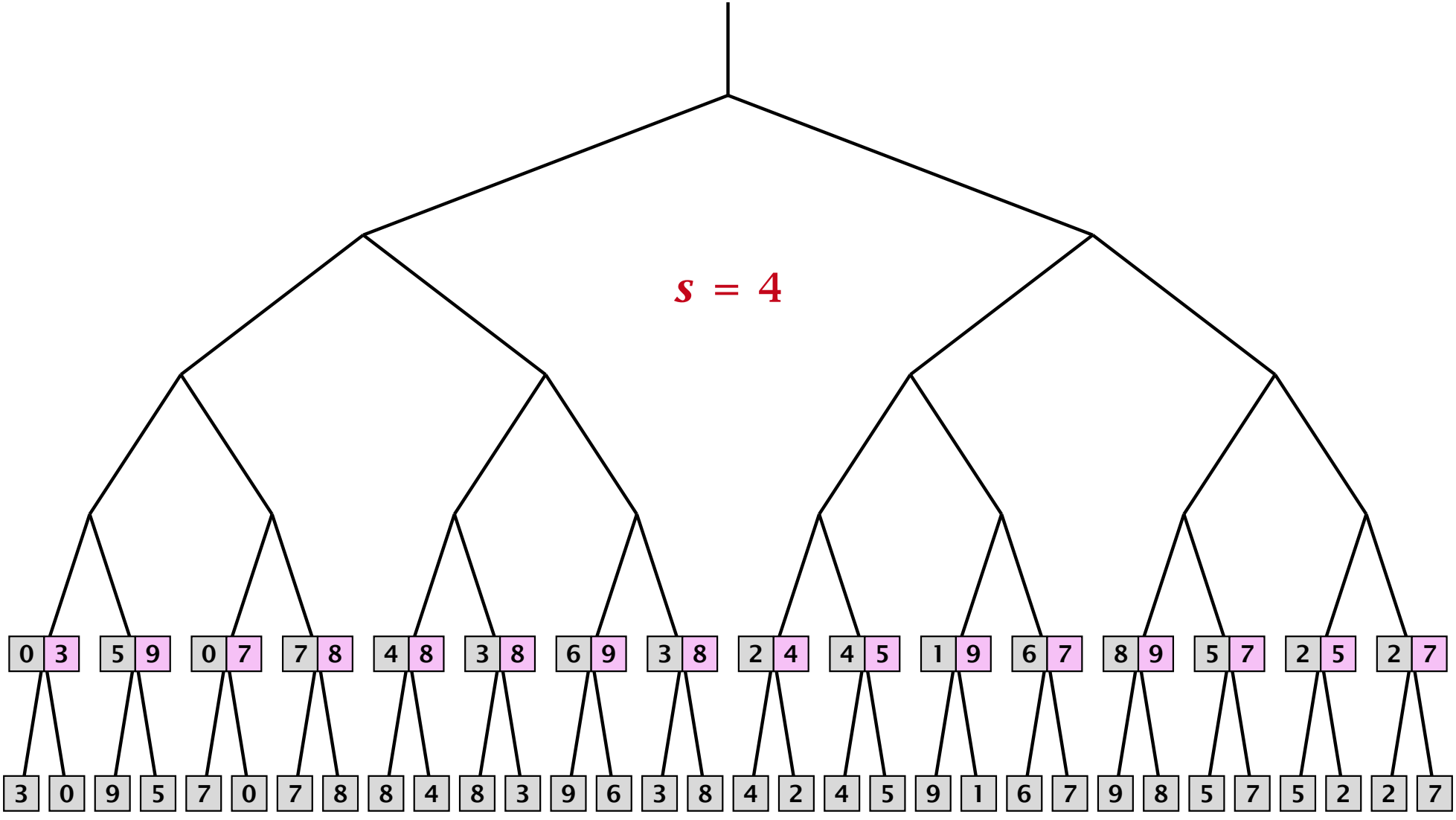
Colesort



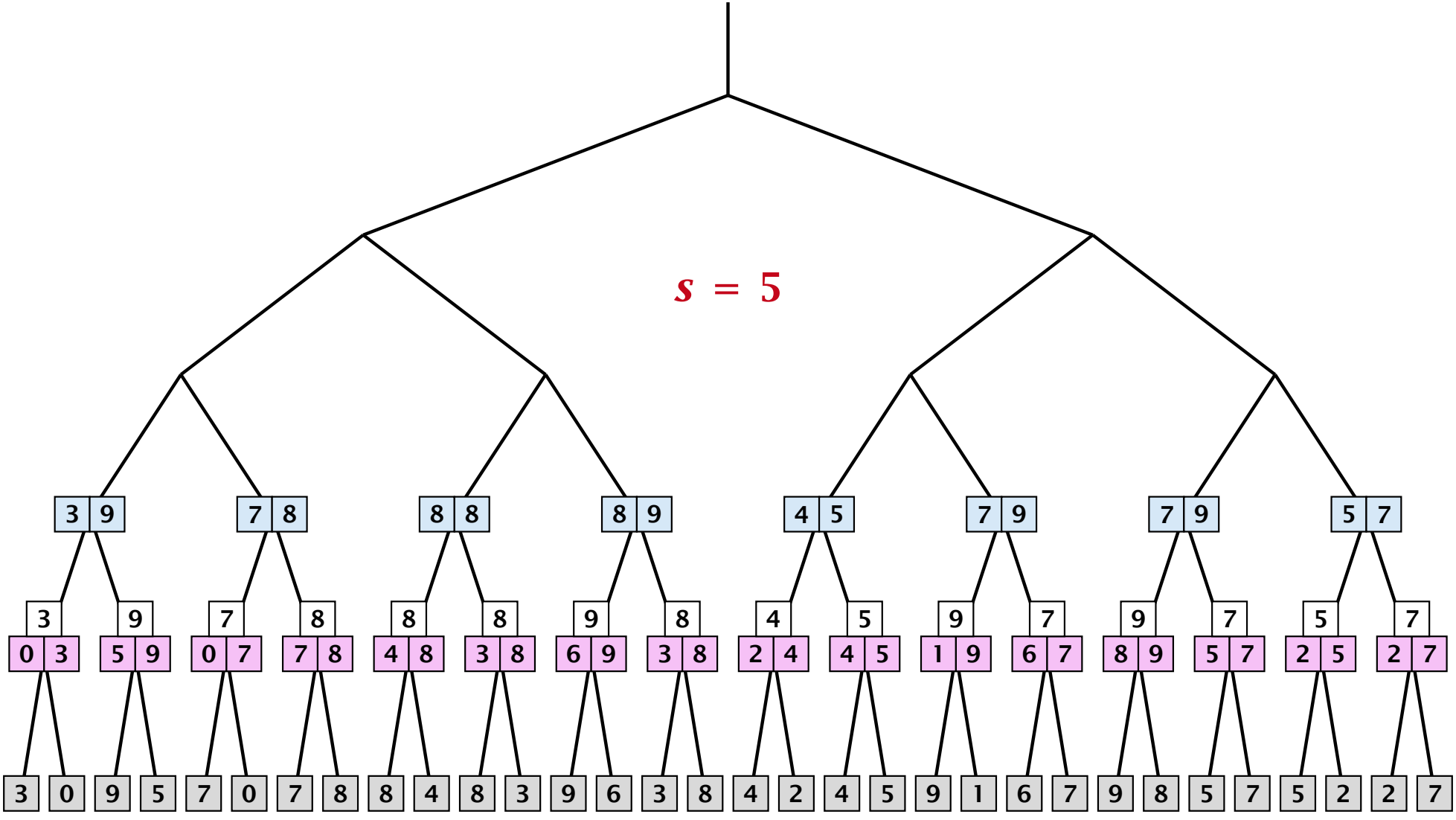
Colesort


s = 3
0 5
2 7
0 8
4 9
3 9
0 6
7 9
0 4
3 7
5 5
3 5
1 4
3 5
1 5
1 2
5 7
0 5 2 7 8 0 4 9 3 9 0 6 7 9 4 0 3 7 5 5 5 3 1 4 3 5 5 1 2 1 5 7
0 5 2 7 8 0 4 9 3 9 0 6 7 9 4 0 3 7 5 5 5 3 1 4 3 5 5 1 2 1 5 7

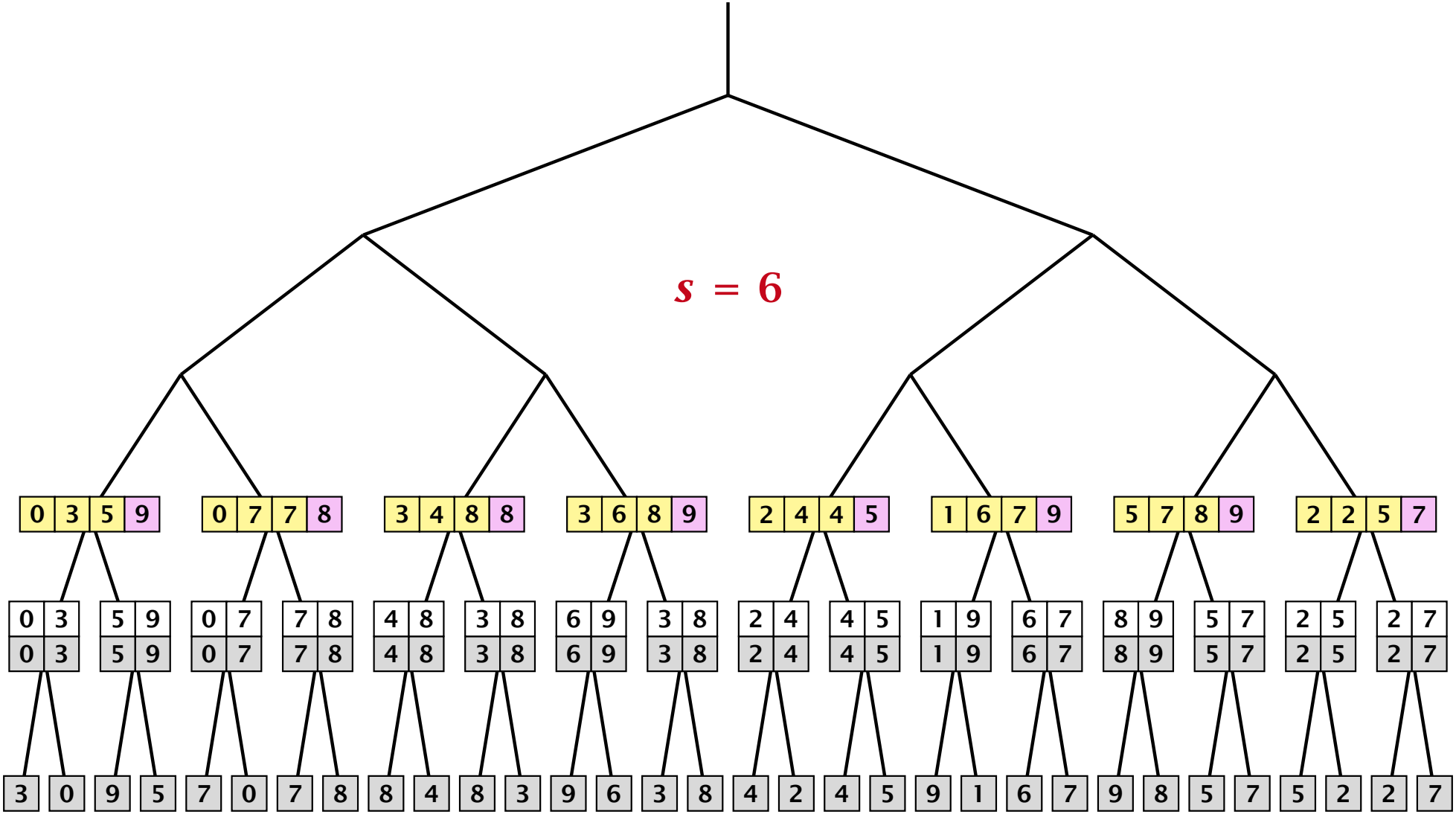
Colesort


$s = 4$
0 3 5 9 0 7 7 8 4 8 3 8 6 9 3 8 2 4 4 5 1 9 6 7 8 9 5 7 2 5 2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

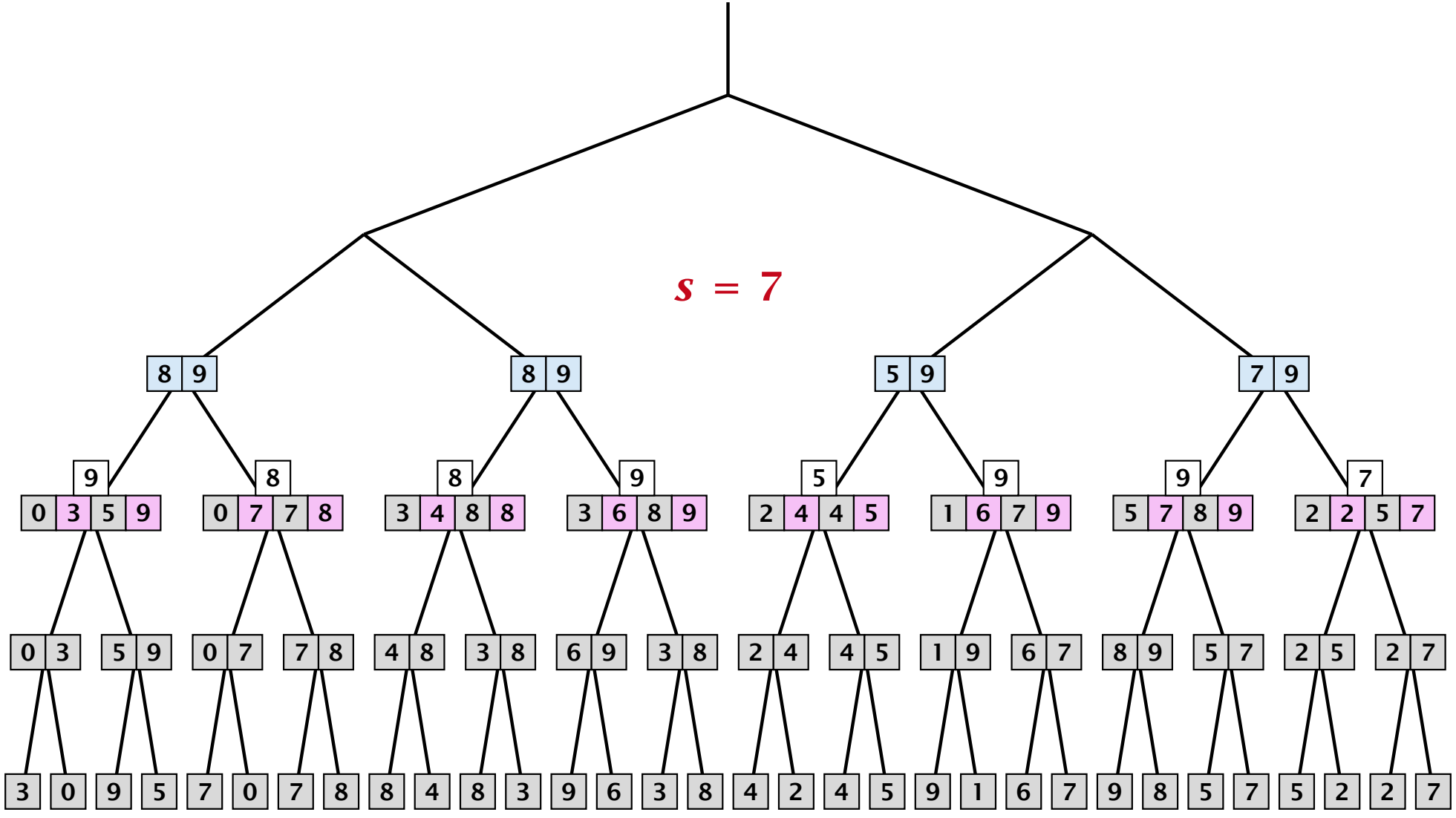
Colesort


s = 5
3 9
7 8
8 8
8 9
4 5
7 9
7 9
5 7
3
9
7
8
8
8
9
8
4
5
9
7
9
7
5
7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

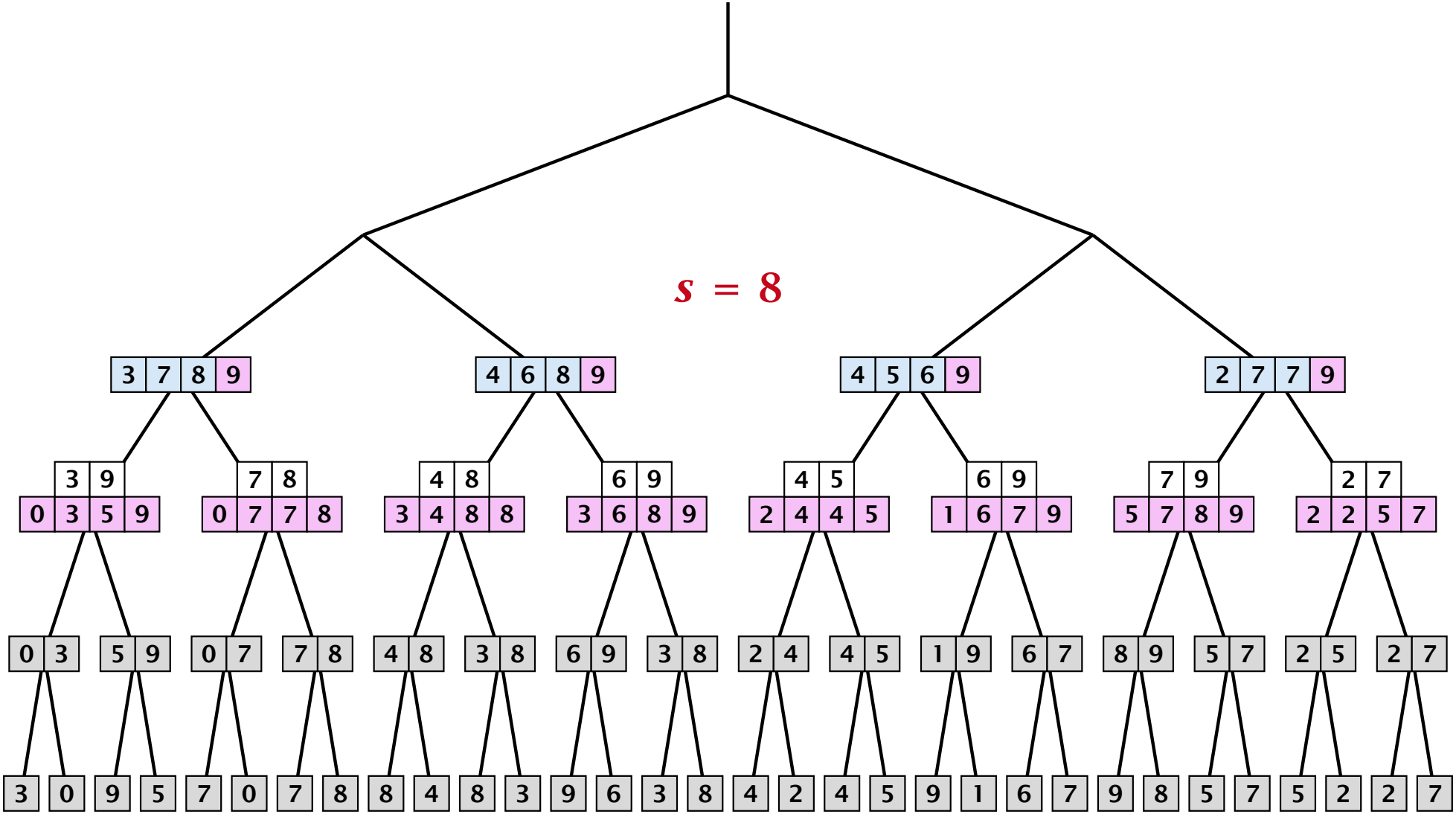
Colesort


s = 6
0 3 5 9
0 7 7 8
3 4 8 8
3 6 8 9
2 4 4 5
1 6 7 9
5 7 8 9
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

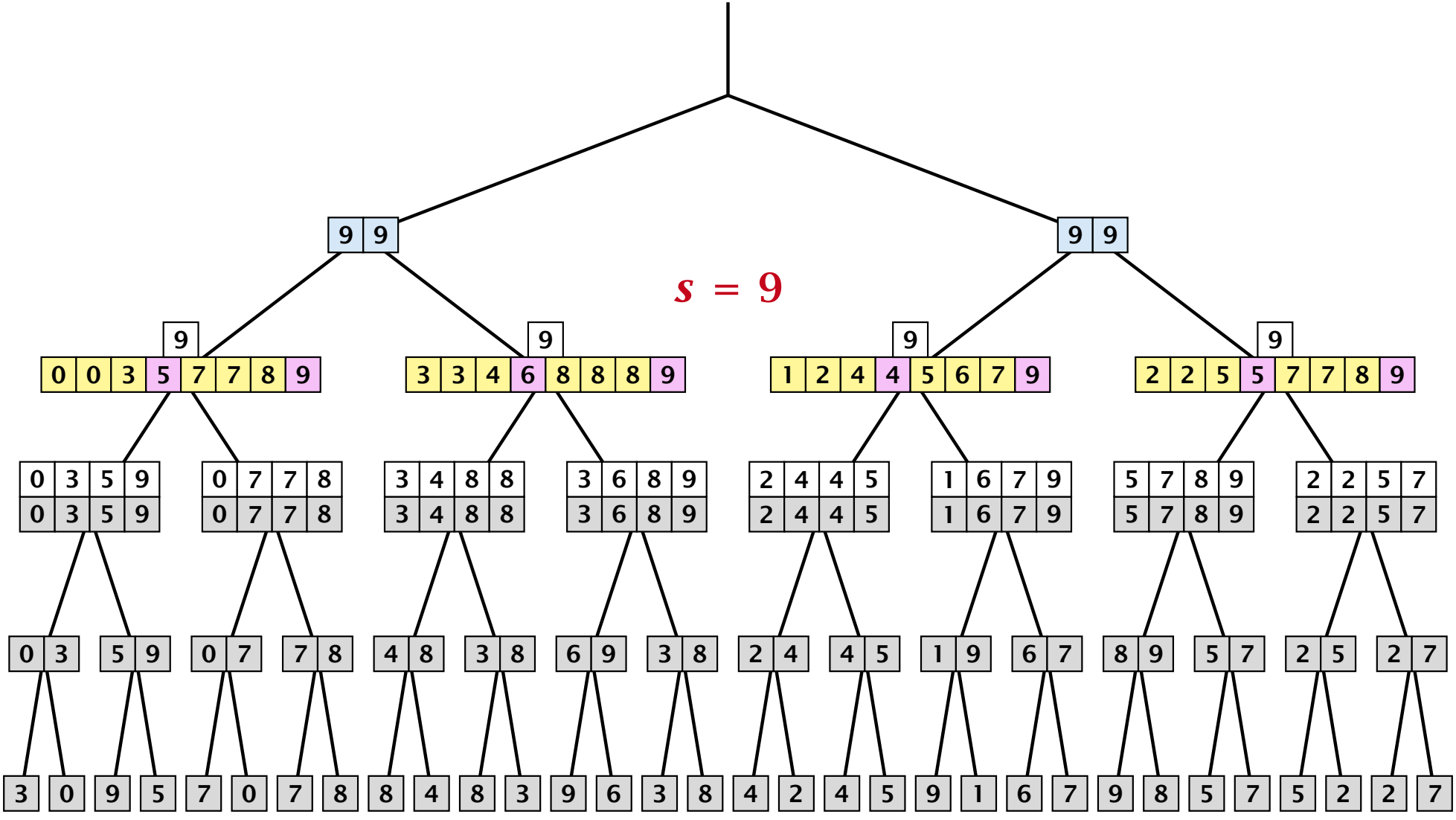
Colesort


s = 7
8 9
8 9
5 9
7 9
9
8
8
9
5
9
9
7
0 3 5 9
0 7 7 8
3 4 8 8
3 6 8 9
2 4 4 5
1 6 7 9
5 7 8 9
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

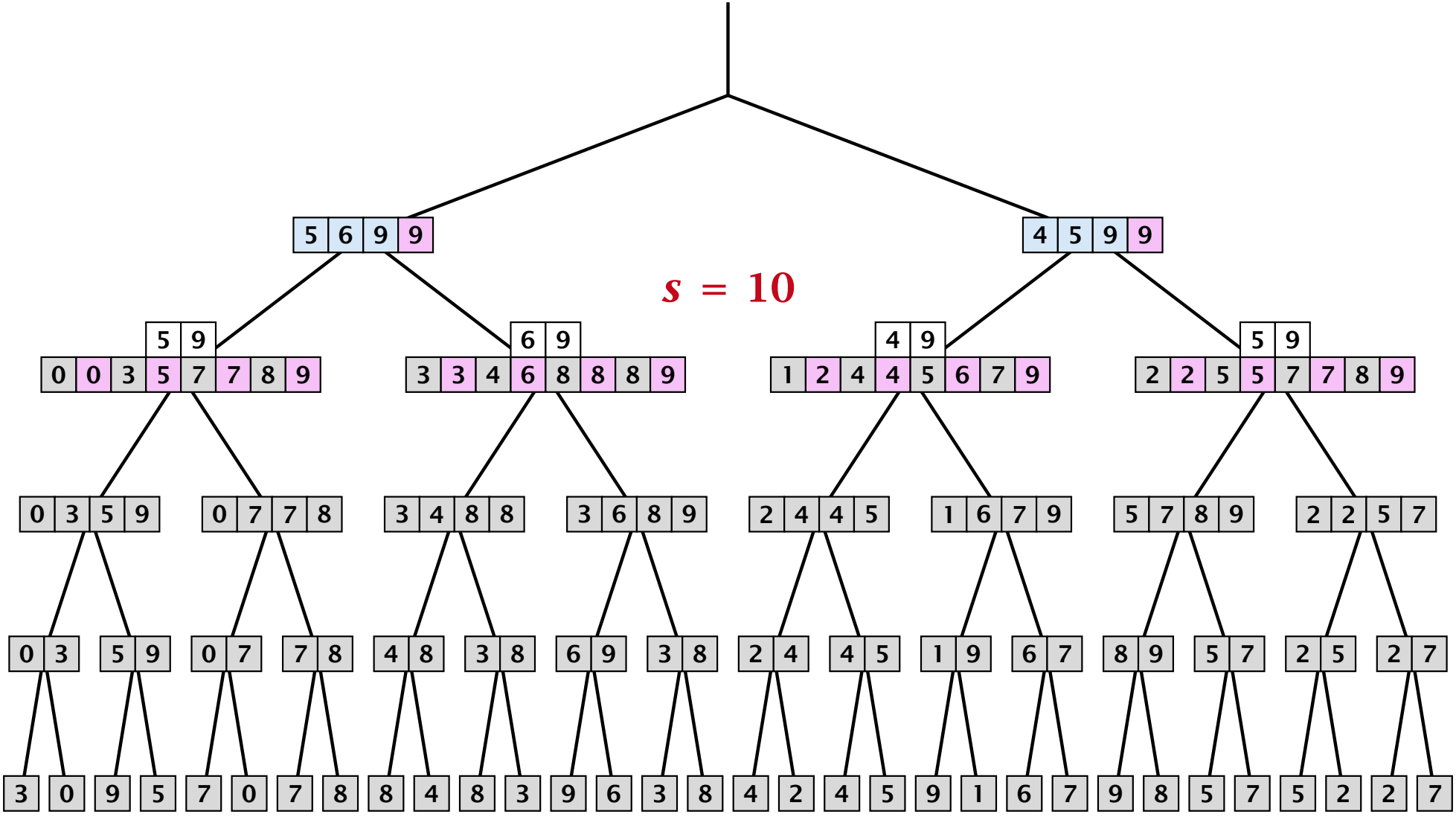
Colesort


s = 8
3 7 8 9
4 6 8 9
4 5 6 9
2 7 7 9
3 9
7 8
4 8
6 9
4 5
6 9
7 9
2 7
0 3 5 9
0 7 7 8
3 4 8 8
3 6 8 9
2 4 4 5
1 6 7 9
5 7 8 9
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

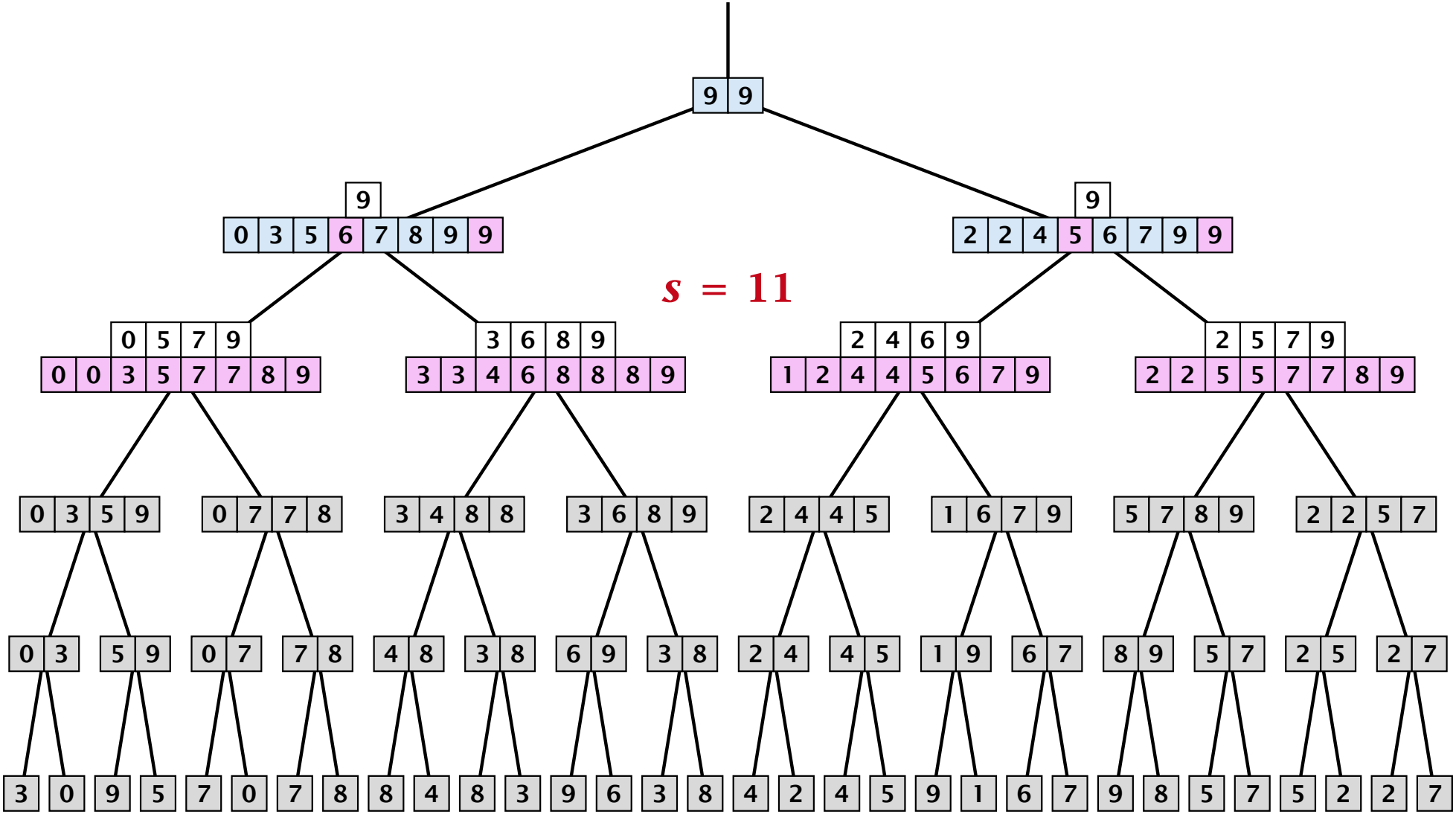
Colesort


s = 9
9 9
9 9
9
0 0 3 5 7 7 8 9
9
3 3 4 6 8 8 8 9
9
1 2 4 4 5 6 7 9
9
2 2 5 5 7 7 8 9
0 3 5 9
0 3 5 9
0 7 7 8
0 7 7 8
3 4 8 8
3 4 8 8
3 6 8 9
3 6 8 9
2 4 4 5
2 4 4 5
1 6 7 9
1 6 7 9
5 7 8 9
5 7 8 9
2 2 5 7
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

Colesort


s = 10
5 6 9 9
4 5 9 9
5 9
0 0 3 5 7 7 8 9
6 9
3 3 4 6 8 8 8 9
4 9
1 2 4 4 5 6 7 9
5 9
2 2 5 5 7 7 8 9
0 3 5 9
0 7 7 8
3 4 8 8
3 6 8 9
2 4 4 5
1 6 7 9
5 7 8 9
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

Colesort


9 9
9
0 3 5 6 7 8 9 9
9
2 2 4 5 6 7 9 9
s = 11
0 5 7 9
0 0 3 5 7 7 8 9
3 6 8 9
3 3 4 6 8 8 8 9
2 4 6 9
1 2 4 4 5 6 7 9
2 5 7 9
2 2 5 5 7 7 8 9
0 3 5 9
0 7 7 8
3 4 8 8
3 6 8 9
2 4 4 5
1 6 7 9
5 7 8 9
2 2 5 7
0 3
5 9
0 7
7 8
4 8
3 8
6 9
3 8
2 4
4 5
1 9
6 7
8 9
5 7
2 5
2 7
3 0 9 5 7 0 7 8 8 4 8 3 9 6 3 8 4 2 4 5 9 1 6 7 9 8 5 7 5 2 2 7

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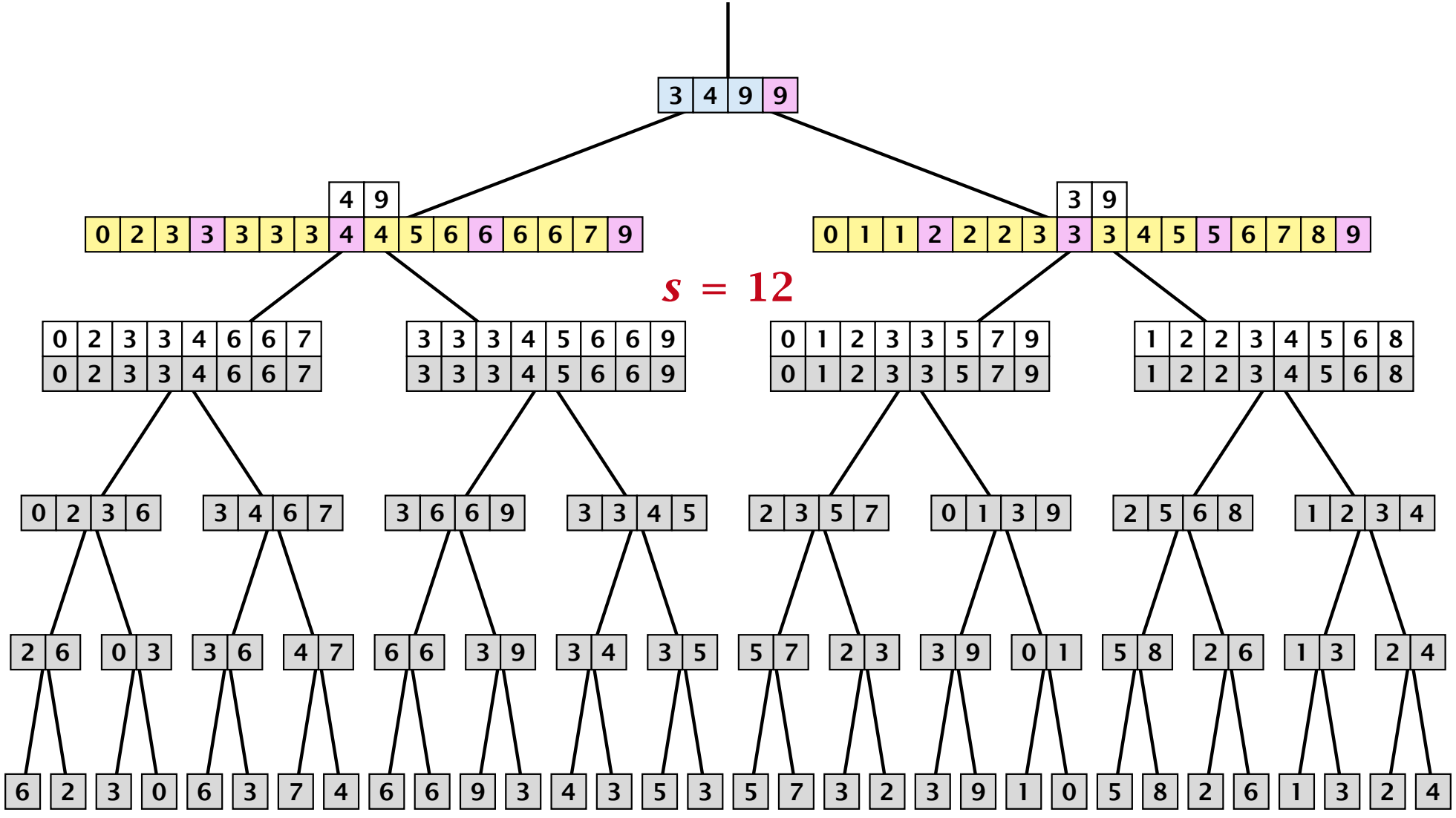
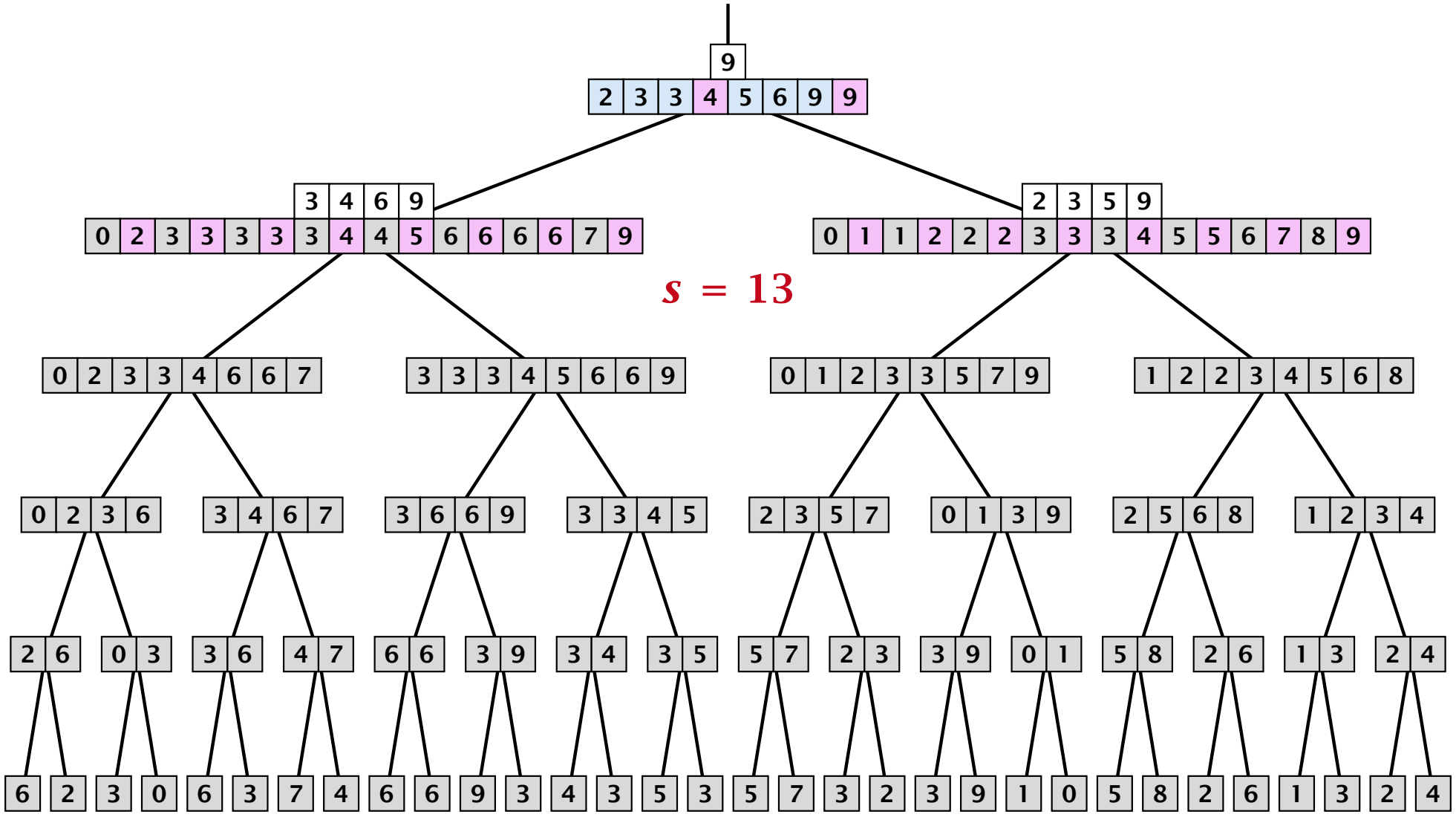
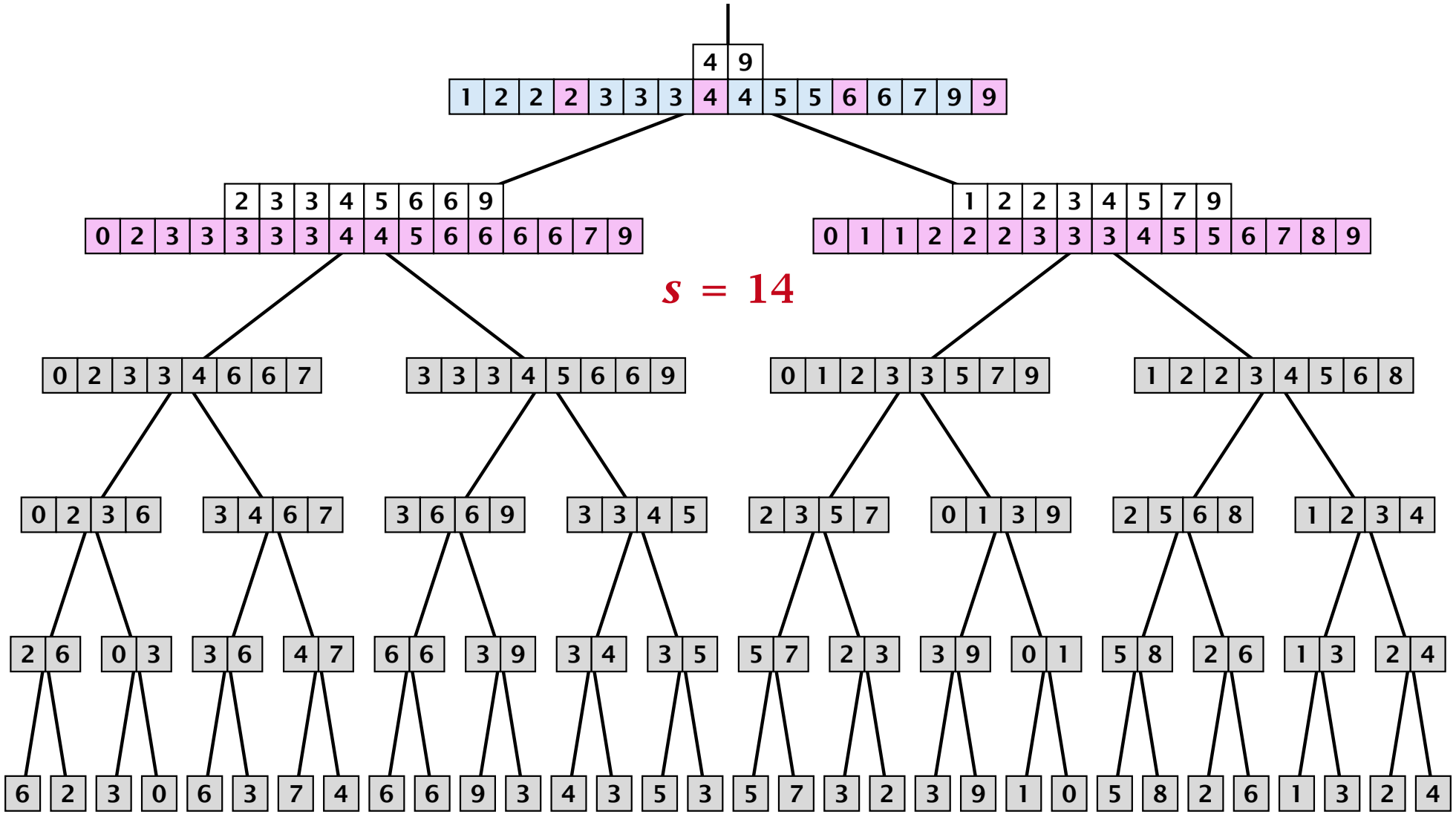

3 4 9 9
4 9
0 2 3 3 3 3 3 4 4 5 6 6 6 6 7 9
3 9
0 1 1 2 2 2 3 3 3 4 5 5 6 7 8 9
s = 12
0 2 3 3 4 6 6 7
0 2 3 3 4 6 6 7
3 3 3 4 5 6 6 9
3 3 3 4 5 6 6 9
0 1 2 3 3 5 7 9
0 1 2 3 3 5 7 9
1 2 2 3 4 5 6 8
1 2 2 3 4 5 6 8
0 2 3 6
3 4 6 7
3 6 6 9
3 3 4 5
2 3 5 7
0 1 3 9
2 5 6 8
1 2 3 4
2 6
0 3
3 6
4 7
6 6
3 9
3 4
3 5
5 7
2 3
3 9
0 1
5 8
2 6
1 3
2 4
6 2 3 0 6 3 7 4 6 6 9 3 4 3 5 3 5 7 3 2 3 9 1 0 5 8 2 6 1 3 2 4

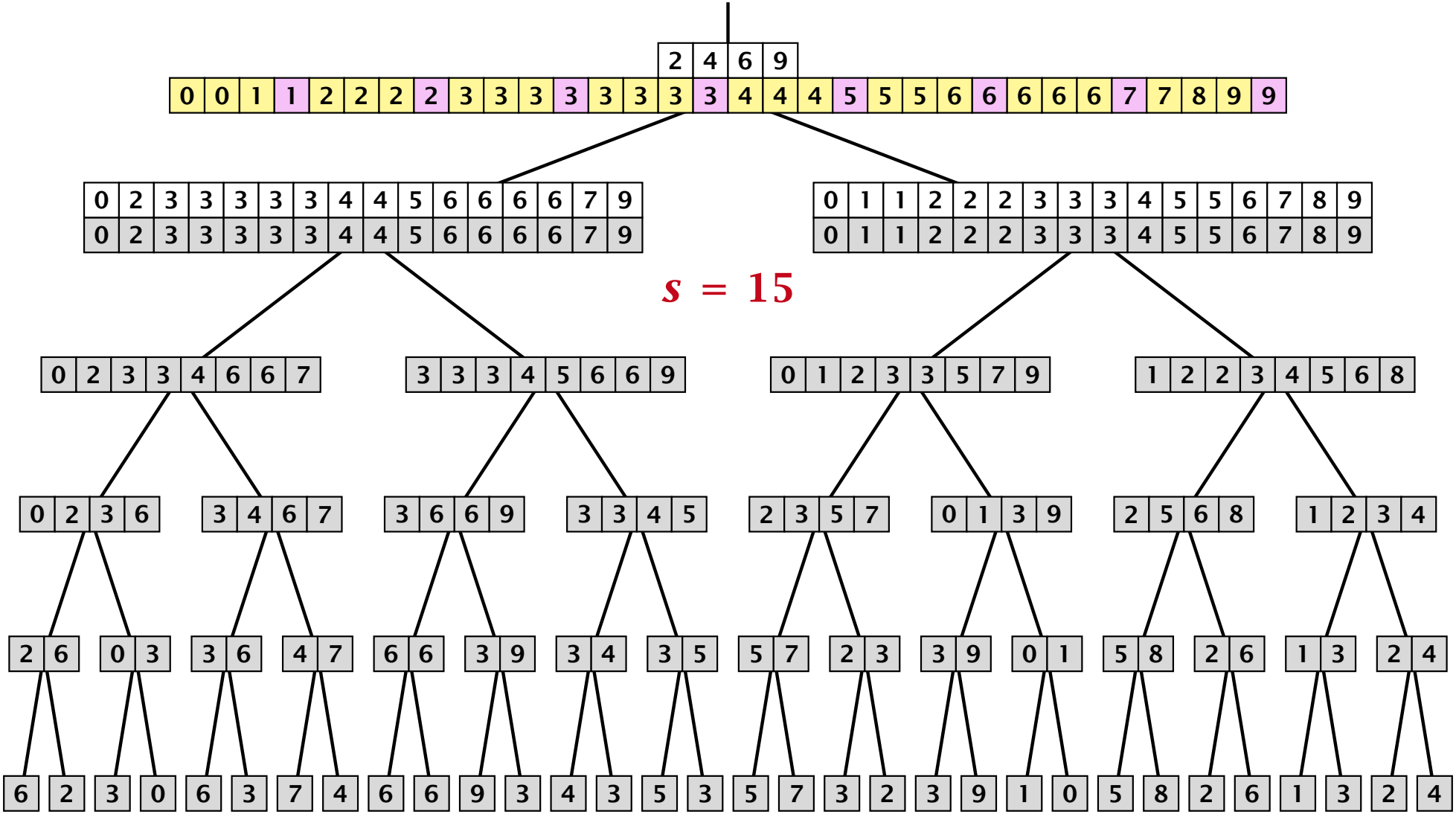
Colesort


9
2 3 3 4 5 6 9 9
3 4 6 9
0 2 3 3 3 3 3 4 4 5 6 6 6 6 7 9
2 3 5 9
0 1 1 2 2 2 3 3 3 4 5 5 6 7 8 9
s = 13
0 2 3 3 4 6 6 7
3 3 3 4 5 6 6 9
0 1 2 3 3 5 7 9
1 2 2 3 4 5 6 8
0 2 3 6
3 4 6 7
3 6 6 9
3 3 4 5
2 3 5 7
0 1 3 9
2 5 6 8
1 2 3 4
2 6
0 3
3 6
4 7
6 6
3 9
3 4
3 5
5 7
2 3
3 9
0 1
5 8
2 6
1 3
2 4
6 2 3 0 6 3 7 4 6 6 9 3 4 3 5 3 5 7 3 2 3 9 1 0 5 8 2 6 1 3 2 4

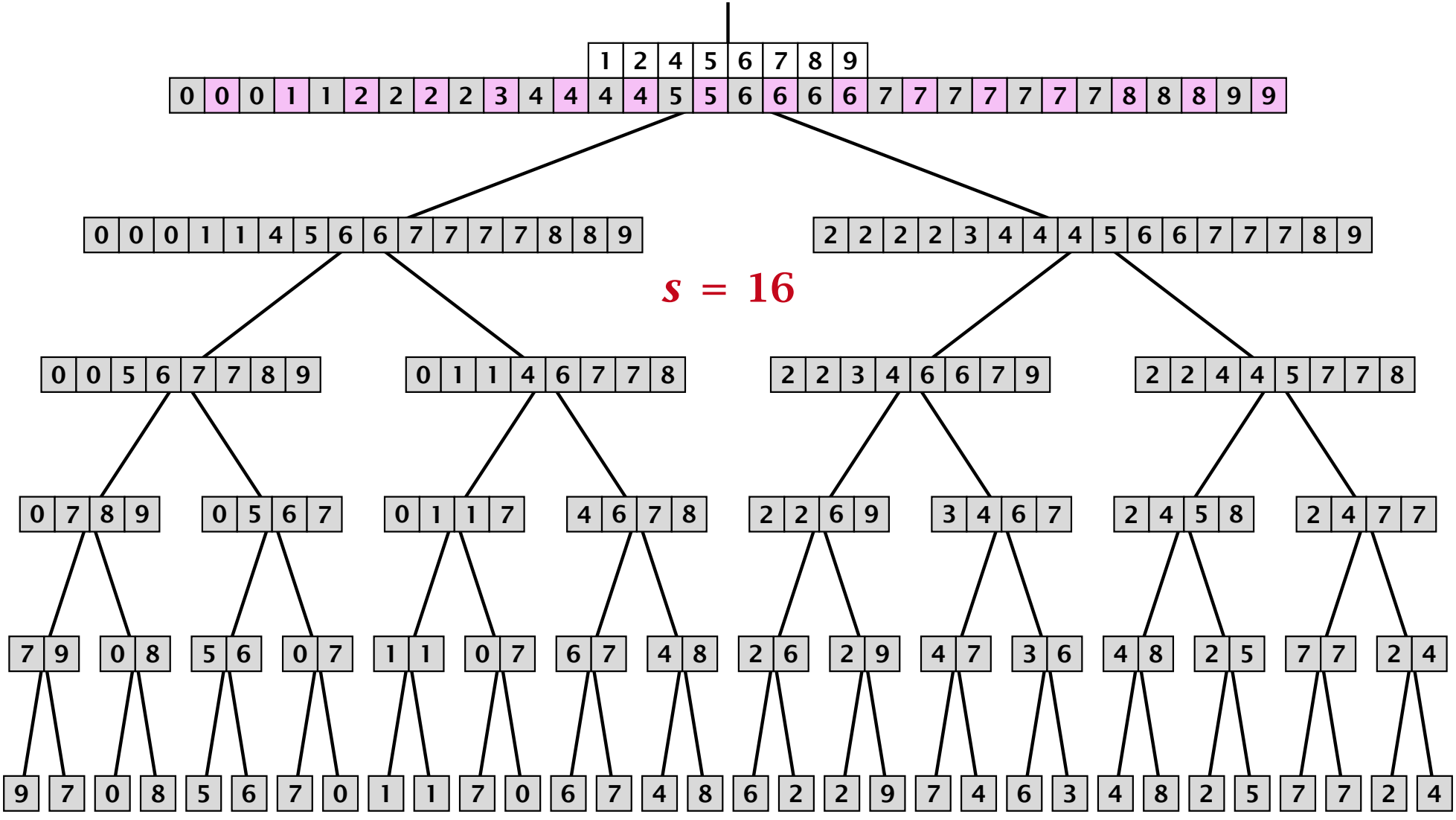
Colesort


4 9
1 2 2 2 3 3 3 4 4 5 5 6 6 7 9 9
2 3 3 4 5 6 6 9
0 2 3 3 3 3 3 4 4 5 6 6 6 6 7 9
1 2 2 3 4 5 7 9
0 1 1 2 2 2 3 3 3 4 5 5 6 7 8 9
s = 14
0 2 3 3 4 6 6 7
3 3 3 4 5 6 6 9
0 1 2 3 3 5 7 9
1 2 2 3 4 5 6 8
0 2 3 6
3 4 6 7
3 6 6 9
3 3 4 5
2 3 5 7
0 1 3 9
2 5 6 8
1 2 3 4
2 6
0 3
3 6
4 7
6 6
3 9
3 4
3 5
5 7
2 3
3 9
0 1
5 8
2 6
1 3
2 4
6 2 3 0 6 3 7 4 6 6 9 3 4 3 5 3 5 7 3 2 3 9 1 0 5 8 2 6 1 3 2 4

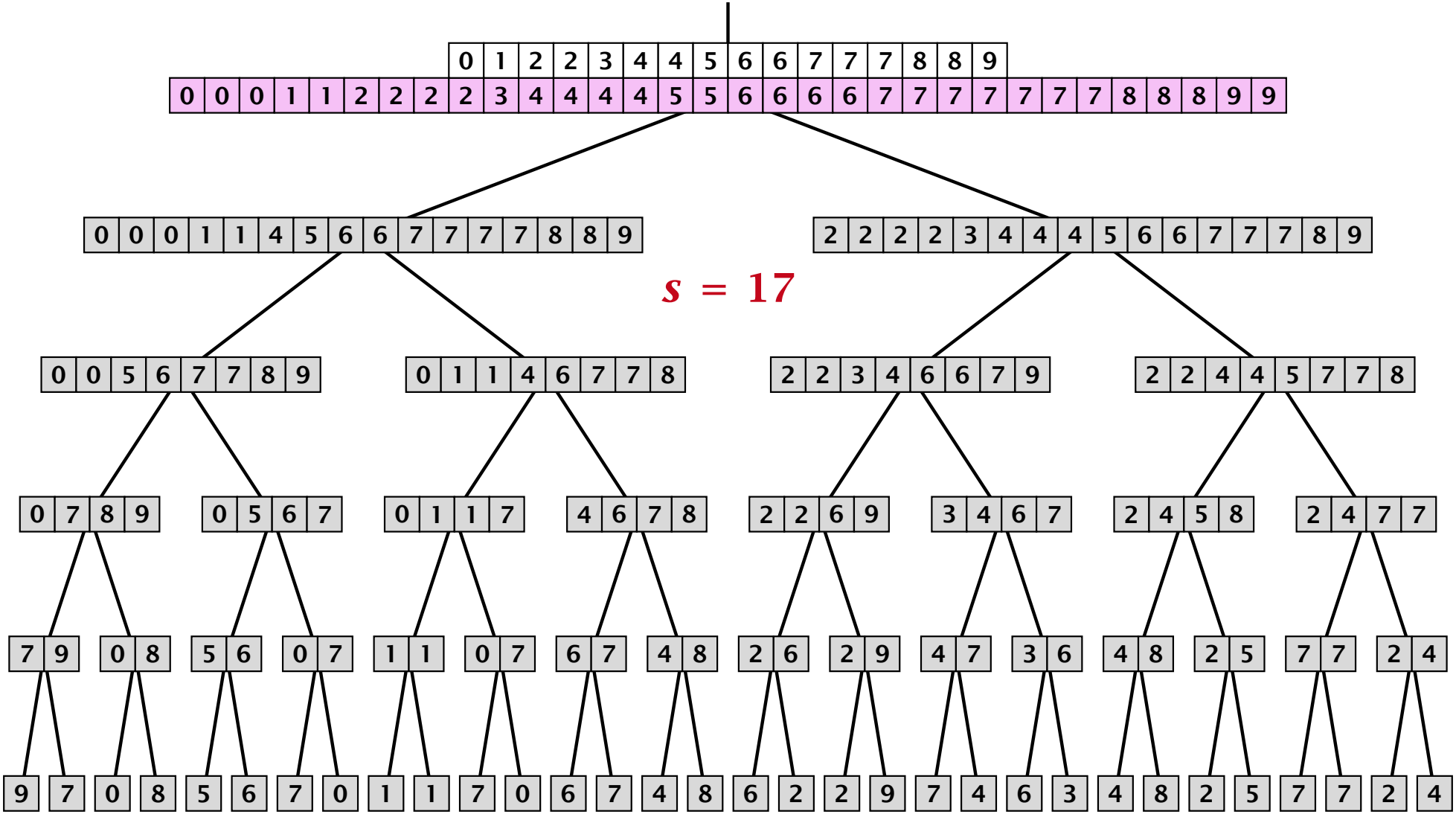
Colesort


2 4 6 9
0 0 1 1 2 2 2 2 3 3 3 3 3 3 3 3 4 4 4 5 5 5 6 6 6 6 6 7 7 8 9 9
0 2 3 3 3 3 3 4 4 5 6 6 6 6 7 9
0 2 3 3 3 3 3 4 4 5 6 6 6 6 7 9
0 1 1 2 2 2 3 3 3 4 5 5 6 7 8 9
0 1 1 2 2 2 3 3 3 4 5 5 6 7 8 9
s = 15
0 2 3 3 4 6 6 7
3 3 3 4 5 6 6 9
0 1 2 3 3 5 7 9
1 2 2 3 4 5 6 8
0 2 3 6
3 4 6 7
3 6 6 9
3 3 4 5
2 3 5 7
0 1 3 9
2 5 6 8
1 2 3 4
2 6
0 3
3 6
4 7
6 6
3 9
3 4
3 5
5 7
2 3
3 9
0 1
5 8
2 6
1 3
2 4
6 2 3 0 6 3 7 4 6 6 9 3 4 3 5 3 5 7 3 2 3 9 1 0 5 8 2 6 1 3 2 4

Colesort


1 2 4 5 6 7 8 9
0 0 0 1 1 2 2 2 2 3 4 4 4 4 5 5 6 6 6 6 7 7 7 7 7 7 7 8 8 8 9 9
0 0 0 1 1 4 5 6 6 7 7 7 7 8 8 9
2 2 2 2 3 4 4 4 5 6 6 7 7 7 8 9
s = 16
0 0 5 6 7 7 8 9
0 1 1 4 6 7 7 8
2 2 3 4 6 6 7 9
2 2 4 4 5 7 7 8
0 7 8 9
0 5 6 7
0 1 1 7
4 6 7 8
2 2 6 9
3 4 6 7
2 4 5 8
2 4 7 7
7 9
0 8
5 6
0 7
1 1
0 7
6 7
4 8
2 6
2 9
4 7
3 6
4 8
2 5
7 7
2 4
9 7 0 8 5 6 7 0 1 1 7 0 6 7 4 8 6 2 2 9 7 4 6 3 4 8 2 5 7 7 2 4

Colesort


0 1 2 2 3 4 4 5 6 6 7 7 7 8 8 9
0 0 0 1 1 2 2 2 2 3 4 4 4 4 5 5 6 6 6 6 7 7 7 7 7 7 7 8 8 8 9 9
0 0 0 1 1 4 5 6 6 7 7 7 7 8 8 9
2 2 2 2 3 4 4 4 5 6 6 7 7 7 8 9
s = 17
0 0 5 6 7 7 8 9
0 1 1 4 6 7 7 8
2 2 3 4 6 6 7 9
2 2 4 4 5 7 7 8
0 7 8 9
0 5 6 7
0 1 1 7
4 6 7 8
2 2 6 9
3 4 6 7
2 4 5 8
2 4 7 7
7 9
0 8
5 6
0 7
1 1
0 7
6 7
4 8
2 6
2 9
4 7
3 6
4 8
2 5
7 7
2 4
9 7 0 8 5 6 7 0 1 1 7 0 6 7 4 8 6 2 2 9 7 4 6 3 4 8 2 5 7 7 2 4

Pipelined Mergesort

Lemma 3

After round $s = 3\,\mathrm{height}(v)$, the list $L_s[v]$ is complete.

Proof:

- clearly true for leaf nodes
- suppose it is true for all nodes up to height h
- fix a node v on level $h+1$ with children z_1 and z_2
- $L_{3h}[z_1]$ and $L_{3h}[z_2]$ are complete by induction hypothesis
- further $\mathrm{sample}(L_{3h+3}[v]) = \mathrm{sample}(L_{3h}[z_1])$ and $\mathrm{sample}(L_{3h+3}[v]) = \mathrm{sample}(L_{3h}[z_2])$
- hence in round $3h+3$ node v will merge the complete list of its children; after the round $L_{3h+3}[v]$ will be complete

Pipelined Mergesort

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- further $\mathrm{sample}(L_{3h+2}[u]) = L[u]$ and $\mathrm{sample}(L_{3h+2}[v]) = L[v]$
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Pipelined Mergesort

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Pipelined Mergesort

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Pipelined Mergesort

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Pipelined Mergesort

Lemma 4

The number of elements in lists $L_s[v]$ for active nodes v is at most $\mathcal{O}(n)$.

proof on board...

Definition 5

A sequence X is a c-cover of a sequence Y if for any two consecutive elements α, β from $(-\infty, X, \infty)$ the set $|\{y_i \mid \alpha \leq y_i \leq \beta\}| \leq c$.

Pipelined Mergesort

Lemma 6

$L'_s[v]$ is a 4-cover of $L'_{s+1}[v]$.

If $[a,b]$ with $a,b \in L'_s[v] \cup \{-\infty,\infty\}$ fulfills $|[a,b] \cap (L'_s[v] \cup \{-\infty,\infty\})| = k$ we say $[a,b]$ intersects $(-\infty, L'_s[v], +\infty)$ in k items.

Lemma 7

If $[a,b]$ intersects $(-\infty, L'_s[v], \infty)$ in $k \geq 2$ items, then $[a,b]$ intersects $(-\infty, L'_{s+1}, \infty)$ in at most $2k$ items.

Pipelined Mergesort

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Pipelined Mergesort

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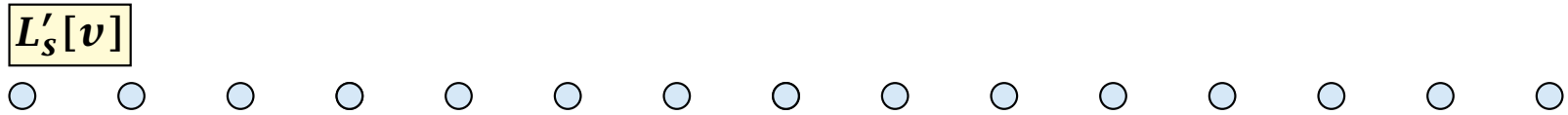
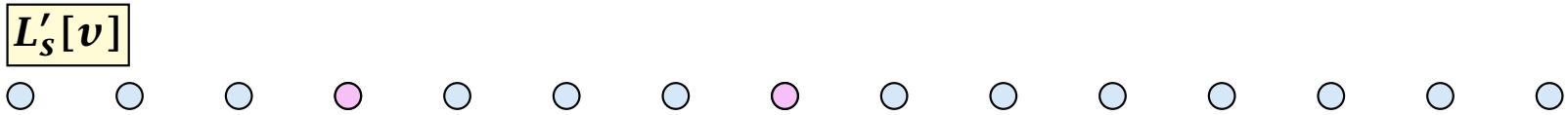
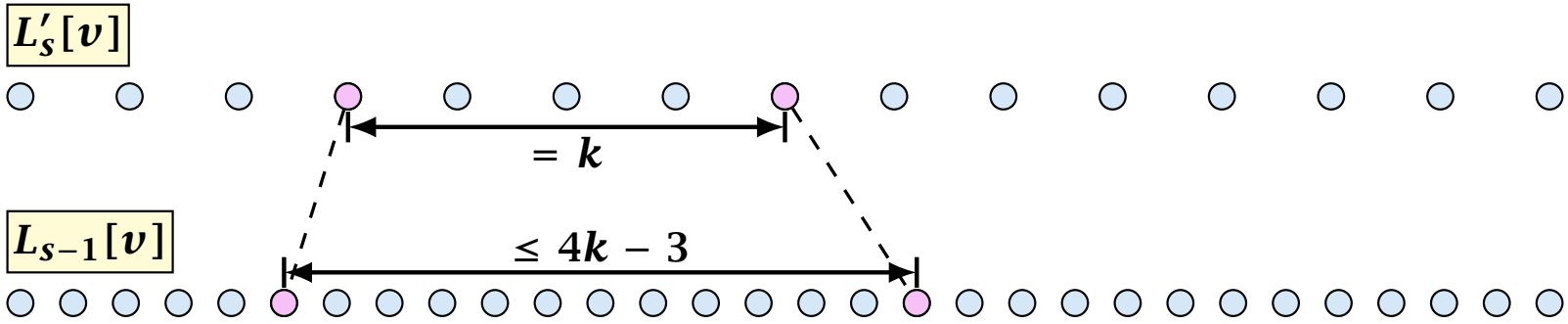
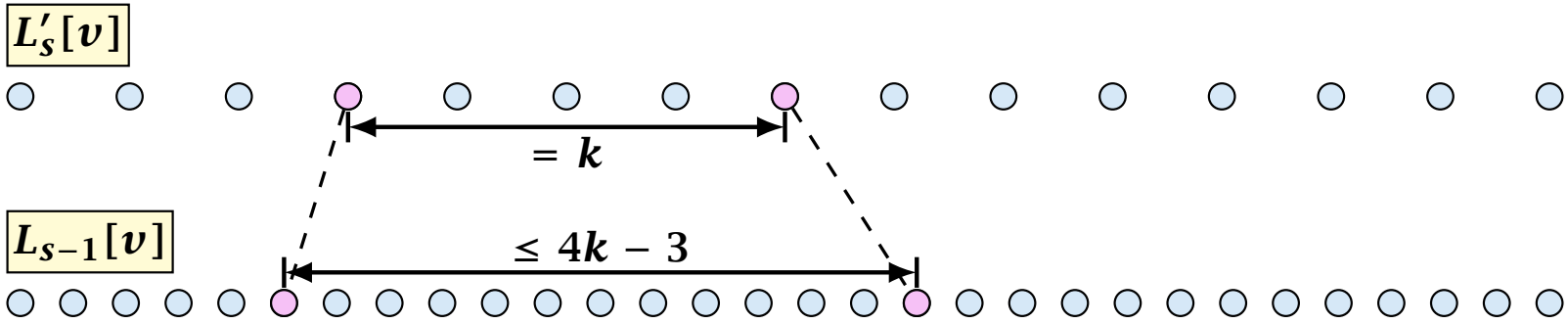

$L'_s[v]$


$L'_s[v]$


$L'_s[v]$
$= k$
$L_{s-1}[v]$
$\leq 4k - 3$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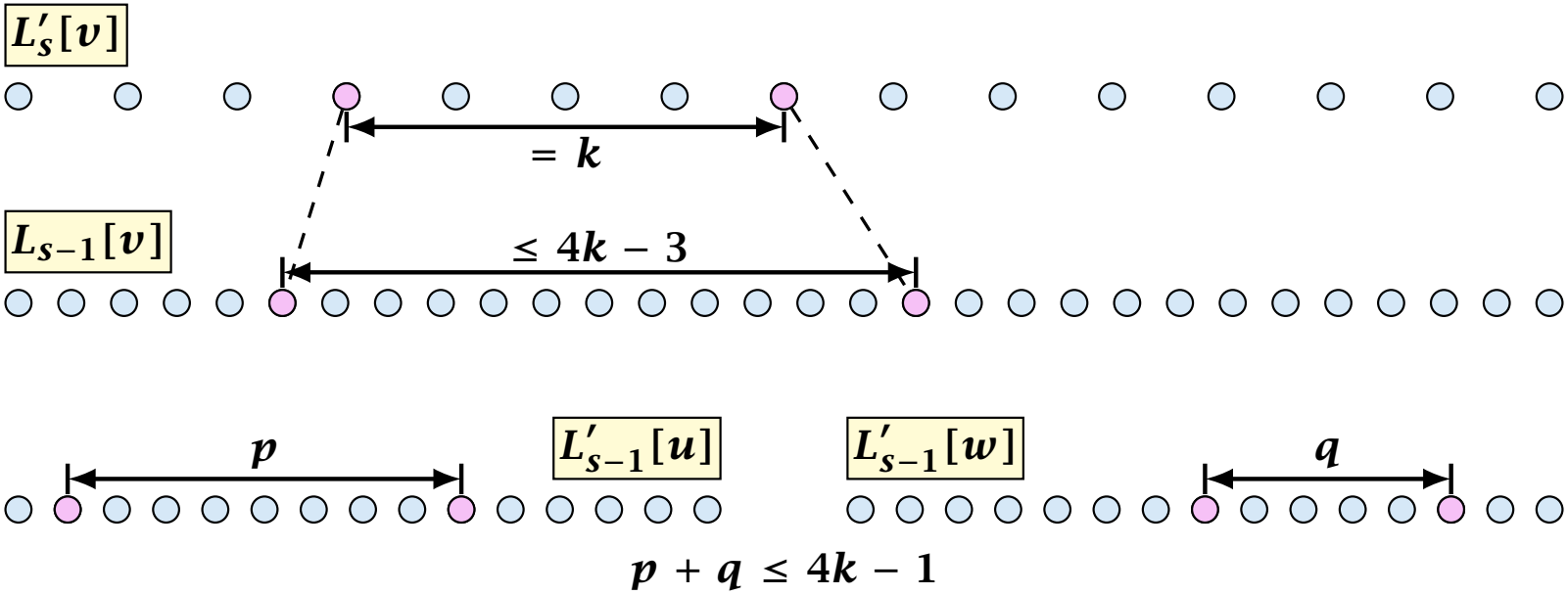
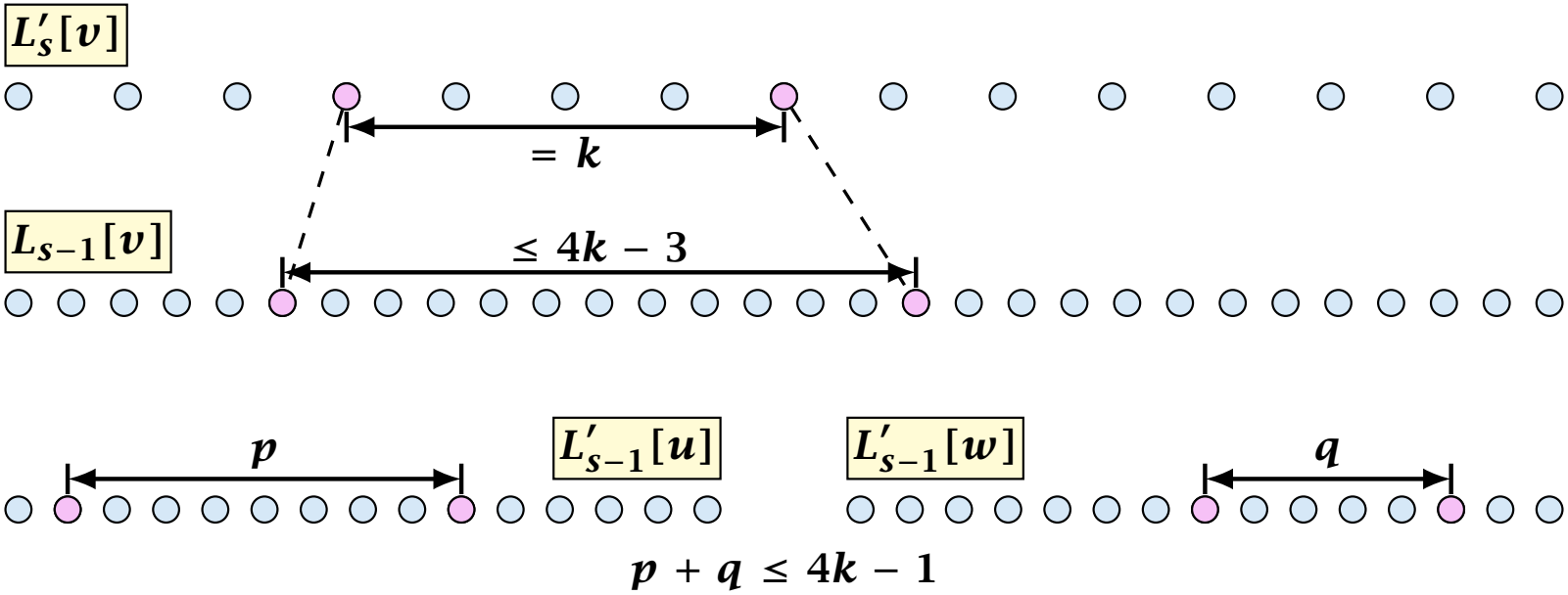


$$p + q \leq 4k - 1$$



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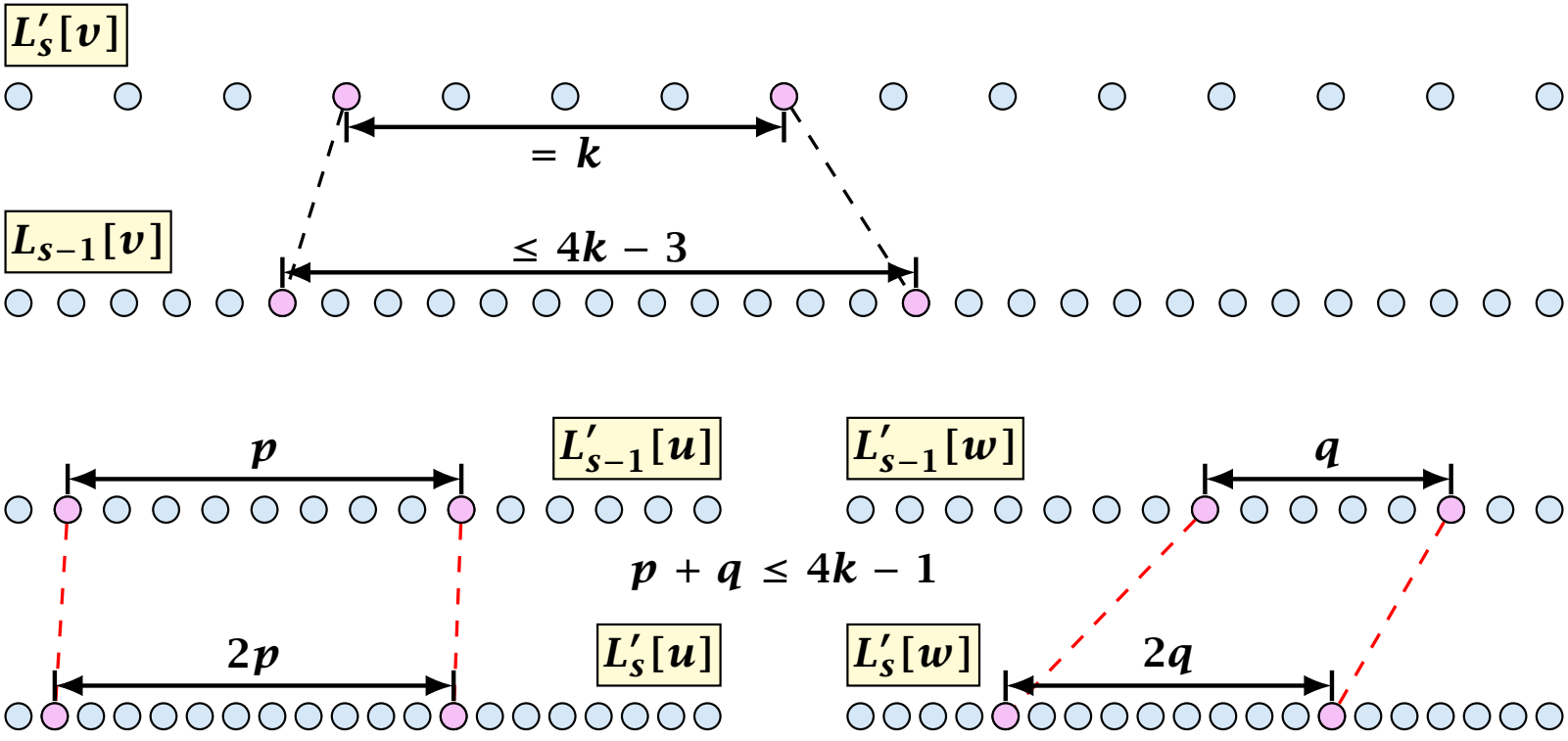
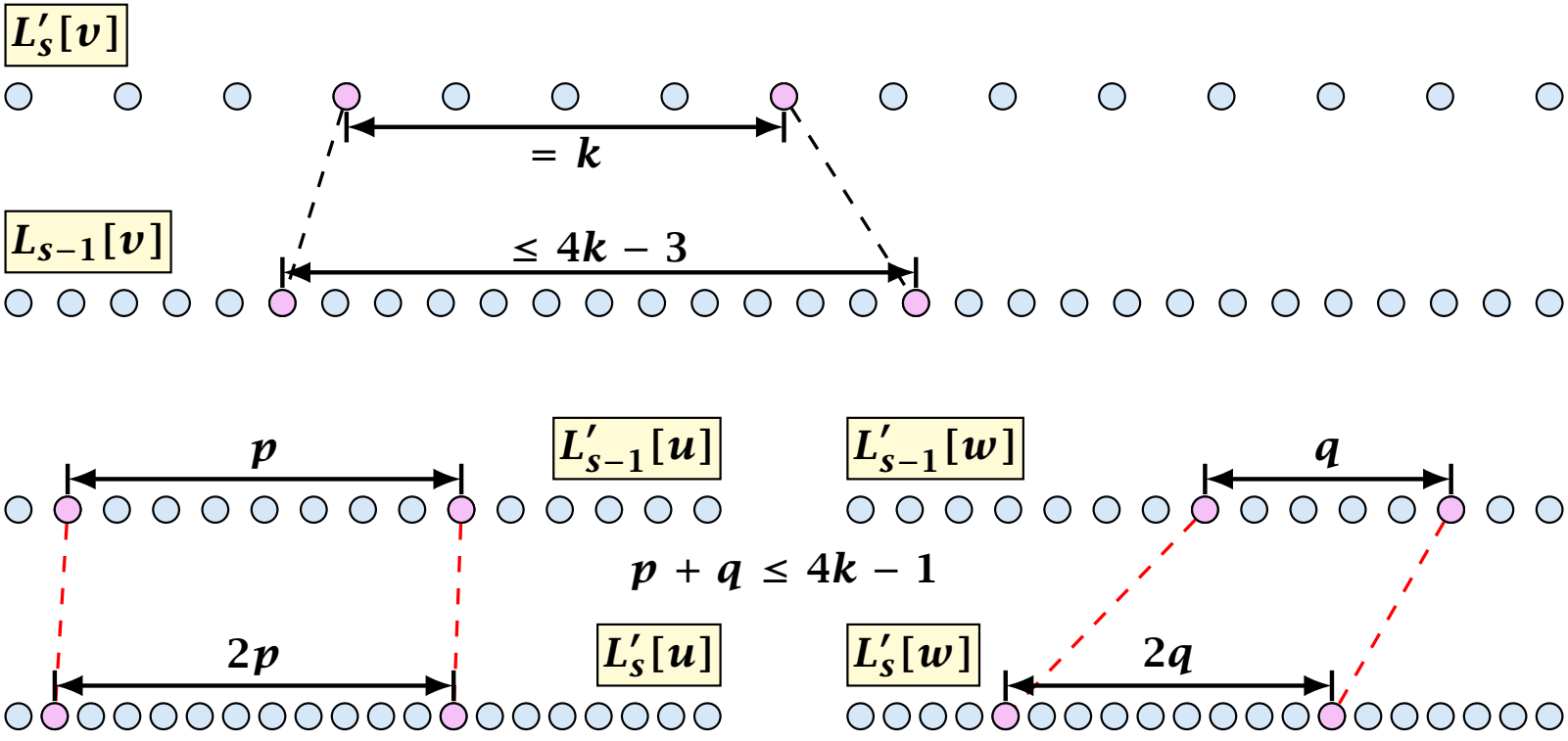

$L'_s[v]$
$= k$
$L_{s-1}[v]$
$\leq 4k - 3$
p
$L'_{s-1}[u]$
$L'_{s-1}[w]$
q
$p + q \leq 4k - 1$
$2p$
$L'_s[u]$
$L'_s[w]$
$2q$


$L'_s[v]$
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q
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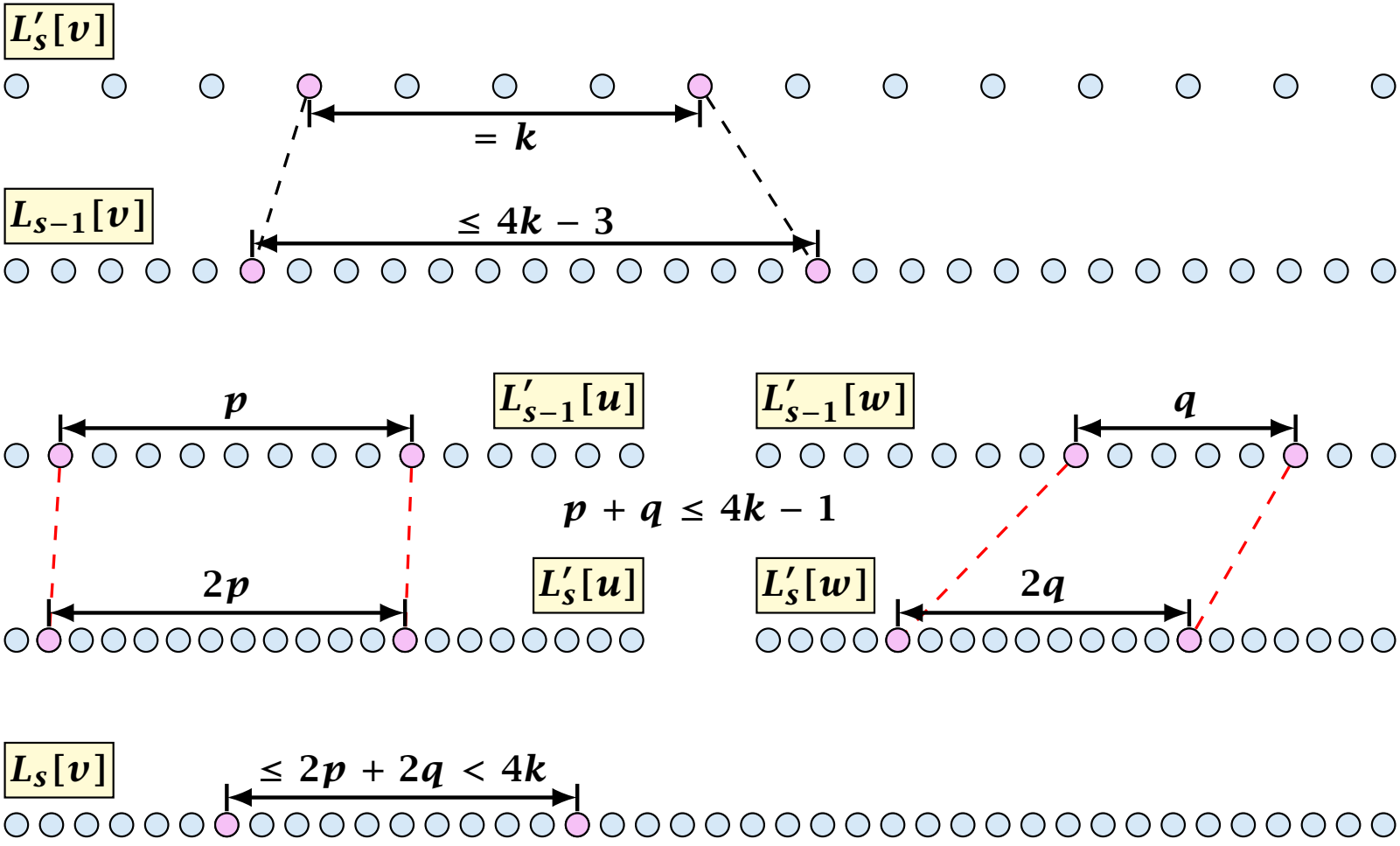
$L'_s[u]$
$L'_s[w]$
$2q$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$\leq 4k - 3$
p
$L'_{s-1}[u]$
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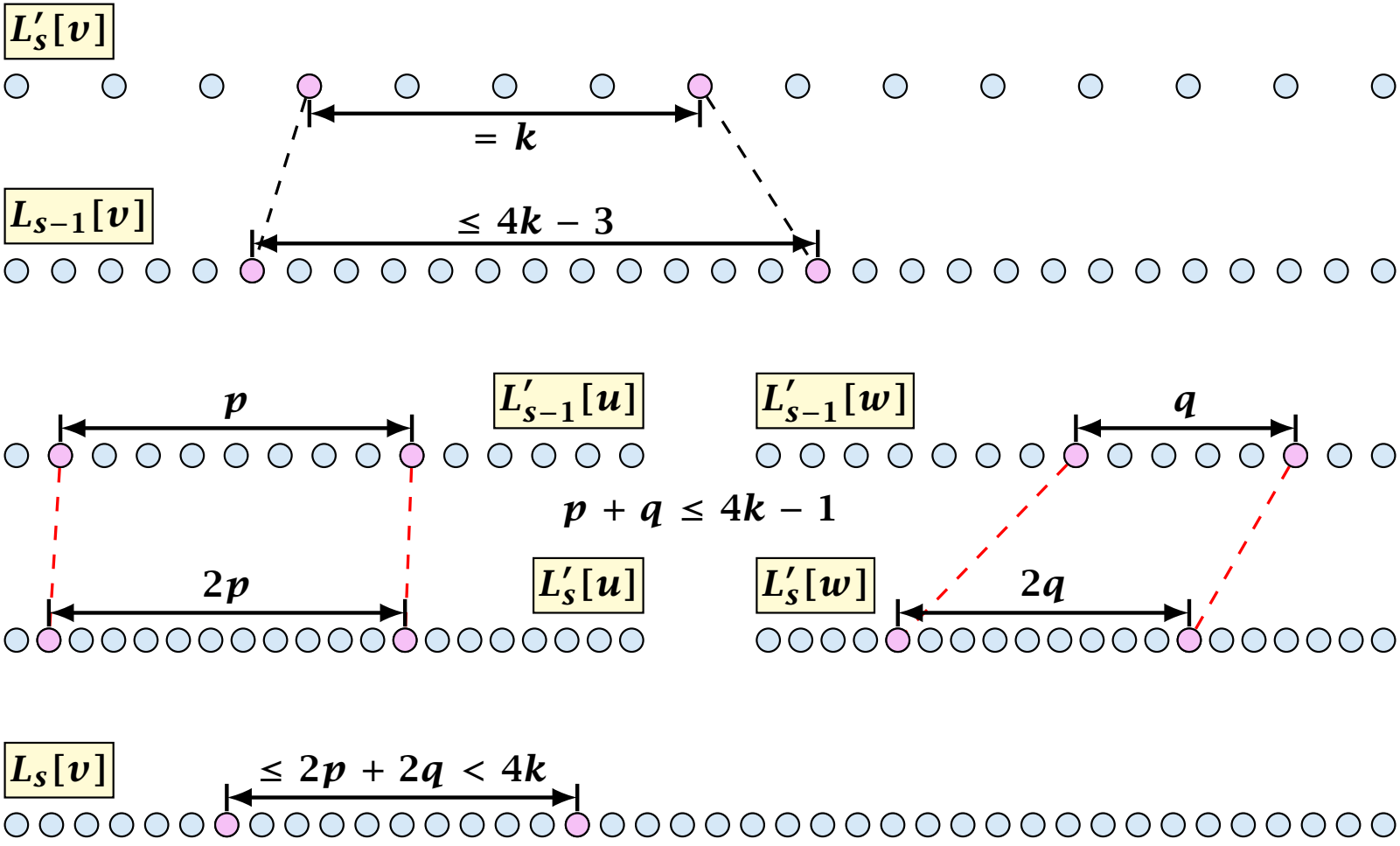
q
$p + q \leq 4k - 1$
$2p$
$L'_s[u]$
$L'_s[w]$
$2q$
$L_s[v]$
$\leq 2p + 2q < 4k$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$L'_{s-1}[u]$
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$2p$
$L'_s[u]$
$L'_s[w]$
$2q$
$L_s[v]$
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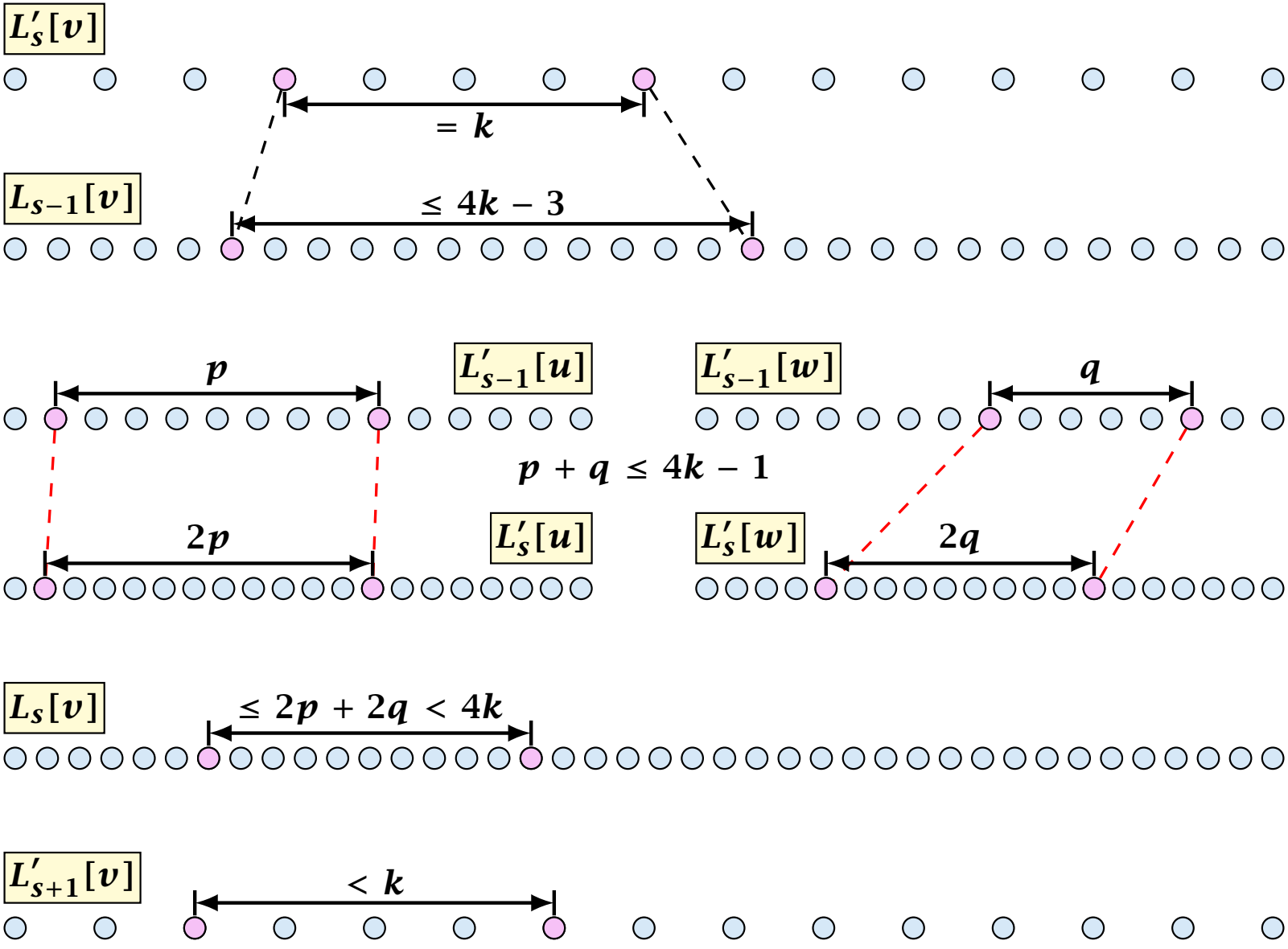
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p
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$p + q \leq 4k - 1$
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$L_s[v]$
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$L'_{s+1}[v]$
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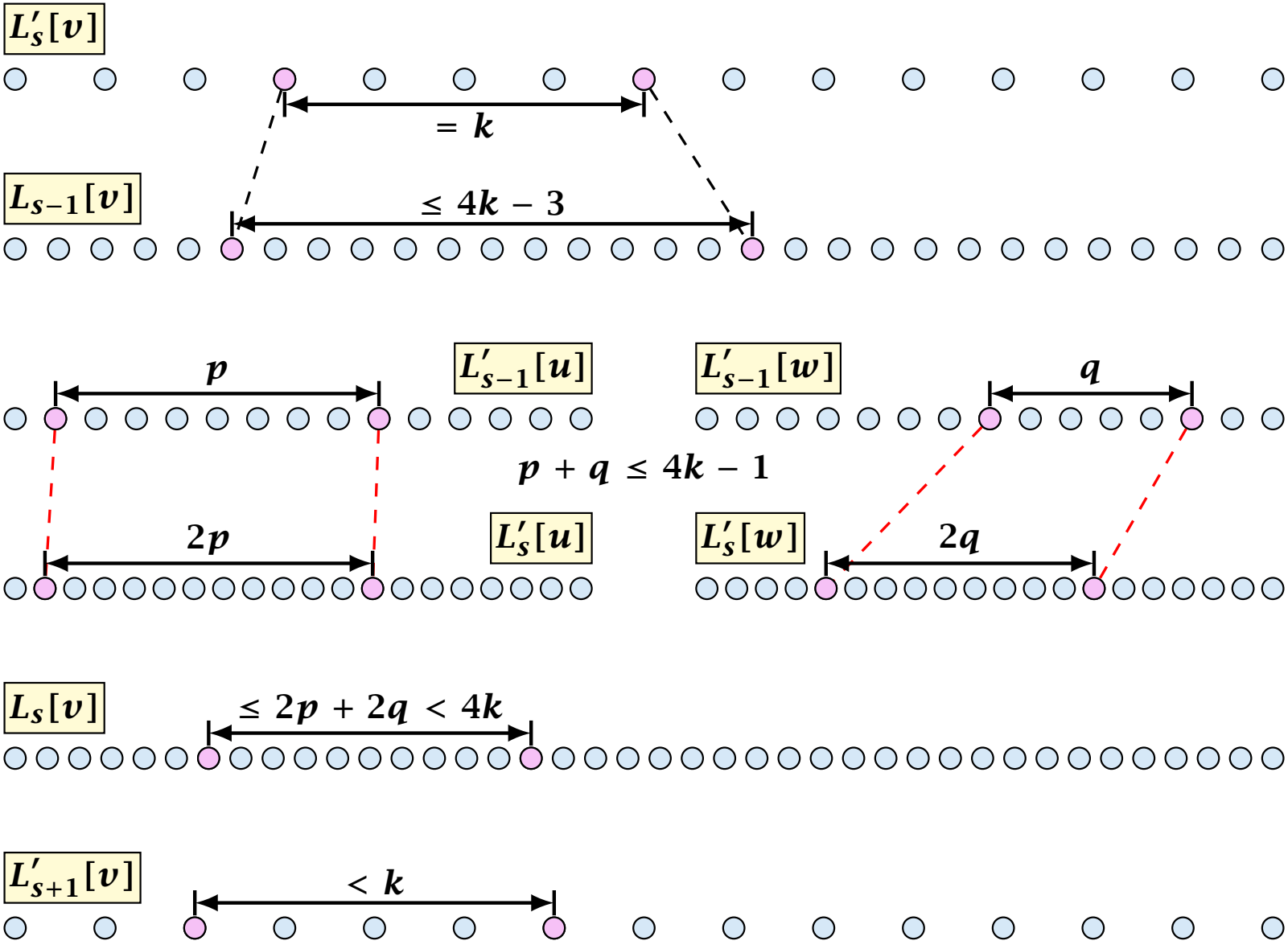
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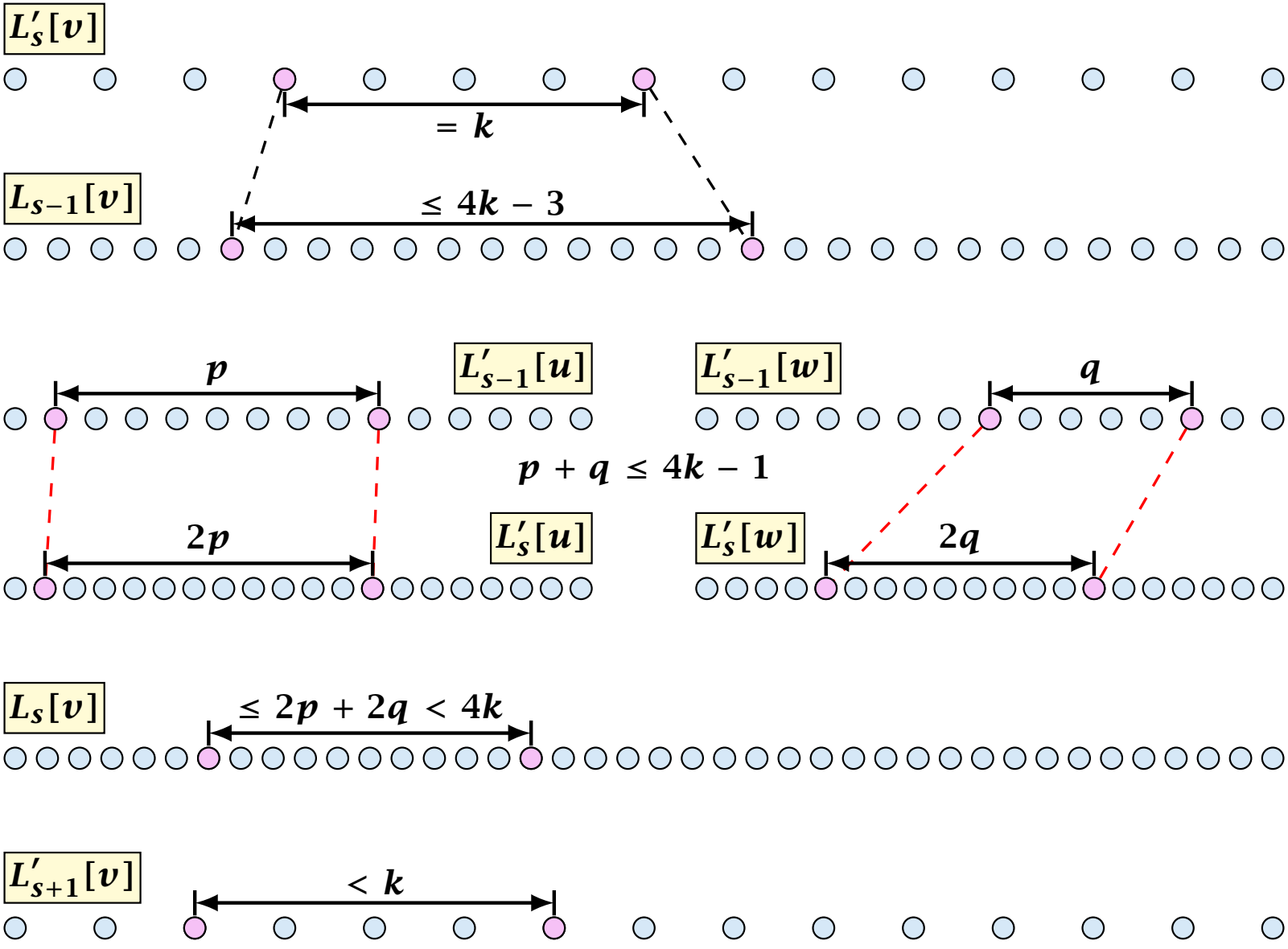
$L'_{s+1}[v]$
$< k$


$L'_s[v]$
$= k$
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p
$L'_{s-1}[u]$
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q
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$2p$
$L'_s[u]$
$L'_s[w]$
$2q$
$L_s[v]$
$\leq 2p + 2q < 4k$
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$< k$

Merging with a Cover

Lemma 8

Given two sorted sequences A and B. Let X be a c-cover of A and B for constant c, and let $\mathrm{rank}(X:A)$ and $\mathrm{rank}(X:B)$ be known.

We can merge A and B in time $\mathcal{O}(1)$ using $\mathcal{O}(|X|)$ operations.

Merging with a Cover

Lemma 9

Given two sorted sequences A and B. Let X be a c-cover of A for constant c, and let $\mathrm{rank}(X:A)$ and $\mathrm{rank}(X:B)$ be known.

We can merge A and B in time $\mathcal{O}(1)$ using $\mathcal{O}(|X|+|B|)$ operations; this means we can compute $\mathrm{rank}(A:B)$ and $\mathrm{rank}(B:A)$.

In order to do the merge in iteration $s+1$ in constant time we need to know

$$\text{rank}(L_s[v] : L'_{s+1}[u]) \text{ and } \text{rank}(L_s[v] : L'_{s+1}[v])$$

and we need to know that $L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[v]$.

Lemma 10

$L_s[v]$ is a 4-cover of $L'_{s+1}[u]$ and $L'_{s+1}[v]$.

- $L_s[v] \supseteq L'_{s+1}[u], L'_{s+1}[v]$
- $L'_{s+1}[u]$ is 4-cover of $L'_{s+1}[u]$
- Hence, $L_s[v]$ is 4-cover of $L'_{s+1}[u]$ as adding more elements cannot destroy the cover-property.

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- Hence, $L_s[v]$ is 4-cover of $L'_{s+1}[u]$ as adding more elements cannot destroy the cover-property.

Analysis

Lemma 11

Suppose we know for every internal node v with children u and w

- $\text{rank}(L'_s[v] : L'_{s+1}[v])$
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$

We can compute

- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$
- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$

in constant time and $\mathcal{O}(|L_{s+1}[v]|)$ operations, where v is the parent of u and w.

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$

ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$

ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_s[w])$
- $\text{rank}(L'_s[w] : L'_s[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute

- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$

Compute

- $\text{rank}(L'_{s+1}[w] : L'_{s+1}[u])$
- $\text{rank}(L'_{s+1}[u] : L'_{s+1}[w])$

ranks between siblings can be computed easily

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute (recall that $L_s[v] = \text{merge}(L'_s[u], L'_s[w])$)

- $\text{rank}(L_s[v] : L'_{s+1}[u])$
- $\text{rank}(L_s[v] : L'_{s+1}[w])$

Compute

- $\text{rank}(L_s[v] : L_{s+1}[v])$ (by adding)
- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$ (by sampling)

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute (recall that $L_s[v] = \text{merge}(L'_s[u], L'_s[w])$)

- $\text{rank}(L_s[v] : L'_{s+1}[u])$
- $\text{rank}(L_s[v] : L'_{s+1}[w])$

Compute

- $\text{rank}(L_s[v] : L_{s+1}[v])$ (by adding)
- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$ (by sampling)

Given

- $\text{rank}(L'_s[u] : L'_{s+1}[u])$ **(4-cover)**
- $\text{rank}(L'_s[u] : L'_{s+1}[w])$
- $\text{rank}(L'_s[w] : L'_{s+1}[u])$
- $\text{rank}(L'_s[w] : L'_{s+1}[w])$ **(4-cover)**

Compute (recall that $L_s[v] = \text{merge}(L'_s[u], L'_s[w])$)

- $\text{rank}(L_s[v] : L'_{s+1}[u])$
- $\text{rank}(L_s[v] : L'_{s+1}[w])$

Compute

- $\text{rank}(L_s[v] : L_{s+1}[v])$ (by adding)
- $\text{rank}(L'_{s+1}[v] : L'_{s+2}[v])$ (by sampling)