How to choose augmenting paths?

- We need to find paths efficiently.

How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

Several possibilities:

How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.

Capacity Scaling

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.

Capacity Scaling

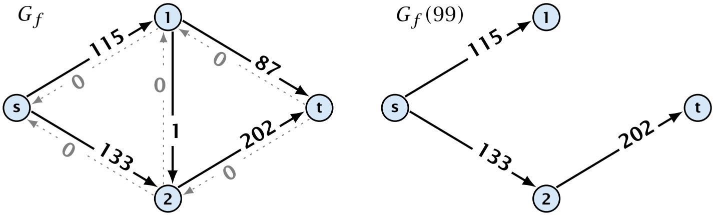
Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.

Capacity Scaling

Intuition:

- Choosing a path with the highest bottleneck increases the flow as much as possible in a single step.
- Don't worry about finding the exact bottleneck.
- Maintain scaling parameter Δ.
- $G_f(\Delta)$ is a sub-graph of the residual graph G_f that contains only edges with capacity at least Δ.



Capacity Scaling

Algorithm 45 maxflow(G, s, t, c)

foreach $e \in E$ **do** $f_e \leftarrow 0$;
$\Delta \leftarrow 2^{\lceil \log_2 C \rceil}$
while $\Delta \geq 1$ **do**
    $G_f(\Delta) \leftarrow \Delta$-residual graph
    while there is augmenting path P in $G_f(\Delta)$ **do**
        $f \leftarrow$ augment(f, c, P)
        update$(G_f(\Delta))$
    $\Delta \leftarrow \Delta/2$
return f

Capacity Scaling

Capacity Scaling

Assumption:

All capacities are integers between 1 and C.

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$

Capacity Scaling

Assumption:
All capacities are integers between 1 and C.

Invariant:
All flows and capacities are/remain integral throughout the algorithm.

Correctness:
The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore

Capacity Scaling

Assumption:

All capacities are integers between 1 and C.

Invariant:

All flows and capacities are/remain integral throughout the algorithm.

Correctness:

The algorithm computes a maxflow:

- because of integrality we have $G_f(1) = G_f$
- therefore after the last phase there are no augmenting paths anymore
- this means we have a maximum flow.

Capacity Scaling

Capacity Scaling

Lemma 1

There are $\lceil \log C \rceil$ *iterations over* Δ*.*

Proof: obvious.

Capacity Scaling

Lemma 1

There are $\lceil \log C \rceil$ *iterations over* Δ.

Proof: obvious.

Lemma 2

Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

Capacity Scaling

Lemma 1

There are $\lceil \log C \rceil$ *iterations over* Δ.

Proof: obvious.

Lemma 2

Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.

Capacity Scaling

Lemma 1

There are $\lceil \log C \rceil$ *iterations over* Δ.

Proof: obvious.

Lemma 2

Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.
- in G_f this cut can have capacity at most $2m\Delta$.

Capacity Scaling

Lemma 1

There are $\lceil \log C \rceil$ *iterations over* Δ.

Proof: obvious.

Lemma 2

Let f *be the flow at the end of a* Δ*-phase. Then the maximum flow is smaller than* $\mathrm{val}(f) + 2m\Delta$.

Proof: less obvious, but simple:

- There must exist an s-t cut in $G_f(\Delta)$ of zero capacity.
- in G_f this cut can have capacity at most $2m\Delta$.
- This gives me an upper bound on the flow that I can still add.

Capacity Scaling

Capacity Scaling

Lemma 3
There are at most $2m$ augmentations per scaling-phase.

Capacity Scaling

Lemma 3

There are at most $2m$ augmentations per scaling-phase.

Proof:

- Let f be the flow at the end of the previous phase.

Capacity Scaling

Lemma 3

There are at most $2m$ augmentations per scaling-phase.

Proof:

- Let f be the flow at the end of the previous phase.
- $\mathrm{val}(f^*) \le \mathrm{val}(f) + 2m\Delta$

Capacity Scaling

Lemma 3

There are at most $2m$ augmentations per scaling-phase.

Proof:

- Let f be the flow at the end of the previous phase.
- $\mathrm{val}(f^*) \le \mathrm{val}(f) + 2m\Delta$
- each augmentation increases flow by Δ.

Capacity Scaling

Lemma 3

There are at most $2m$ augmentations per scaling-phase.

Proof:

- Let f be the flow at the end of the previous phase.
- $\mathrm{val}(f^*) \le \mathrm{val}(f) + 2m\Delta$
- each augmentation increases flow by Δ.

Theorem 4

We need $\mathcal{O}(m \log C)$ augmentations. The algorithm can be implemented in time $\mathcal{O}(m^2 \log C)$.