7.3 AVL-Trees

Definition 1

AVL-trees are binary search trees that fulfill the following balance condition. For every node v

$$|\text{height}(\text{left sub-tree}(v)) - \text{height}(\text{right sub-tree}(v))| \leq 1 \ .$$

Lemma 2

An AVL-tree of height h contains at least $F_{h+2} - 1$ and at most $2^h - 1$ internal nodes, where F_n is the n-th Fibonacci number ($F_0 = 0$, $F_1 = 1$), and the height is the maximal number of edges from the root to an (empty) dummy leaf.

Proof.

The upper bound is clear, as a binary tree of height h can only contain

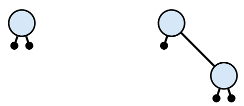
$$\sum_{j=0}^{h-1} 2^j = 2^h - 1$$

internal nodes.

Proof (cont.)

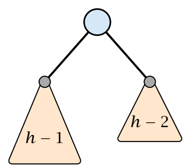
Induction (base cases):

1. an AVL-tree of height $h = 1$ contains at least one internal node, $1 \geq F_3 - 1 = 2 - 1 = 1$.
2. an AVL tree of height $h = 2$ contains at least two internal nodes, $2 \geq F_4 - 1 = 3 - 1 = 2$



Induction step:

An AVL-tree of height $h \geq 2$ of minimal size has a root with sub-trees of height $h - 1$ and $h - 2$, respectively. Both, sub-trees have minmal node number.



Let

$$g_h := 1 + \text{minimal size of AVL-tree of height } h \ .$$

Then

$$\begin{aligned} g_1 &= 2 && = F_3 \\ g_2 &= 3 && = F_4 \\ g_h - 1 &= 1 + g_{h-1} - 1 + g_{h-2} - 1 \,, && \text{hence} \\ g_h &= g_{h-1} + g_{h-2} && = F_{h+2} \end{aligned}$$

7.3 AVL-Trees

An AVL-tree of height h contains at least $F_{h+2} - 1$ internal nodes. Since

$$n + 1 \geq F_{h+2} = \Omega\left(\left(\frac{1+\sqrt{5}}{2}\right)^h\right) \ ,$$

we get

$$n \geq \Omega\left(\left(\frac{1+\sqrt{5}}{2}\right)^h\right) \ ,$$

and, hence, $h = \mathcal{O}(\log n)$.

7.3 AVL-Trees

We need to maintain the balance condition through rotations.

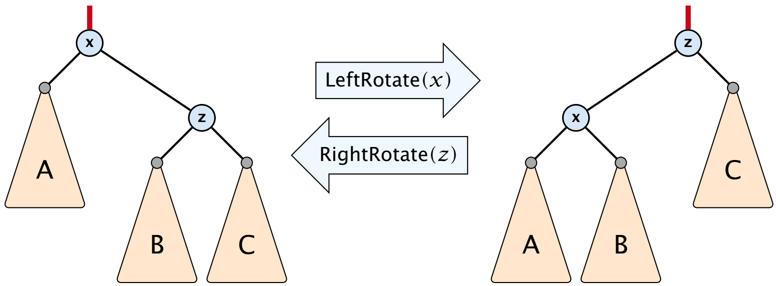
For this we store in every internal tree-node v the balance of the node. Let v denote a tree node with left child c_ℓ and right child c_r.

$$\text{balance}[v] := \text{height}(T_{c_\ell}) - \text{height}(T_{c_r}) \ ,$$

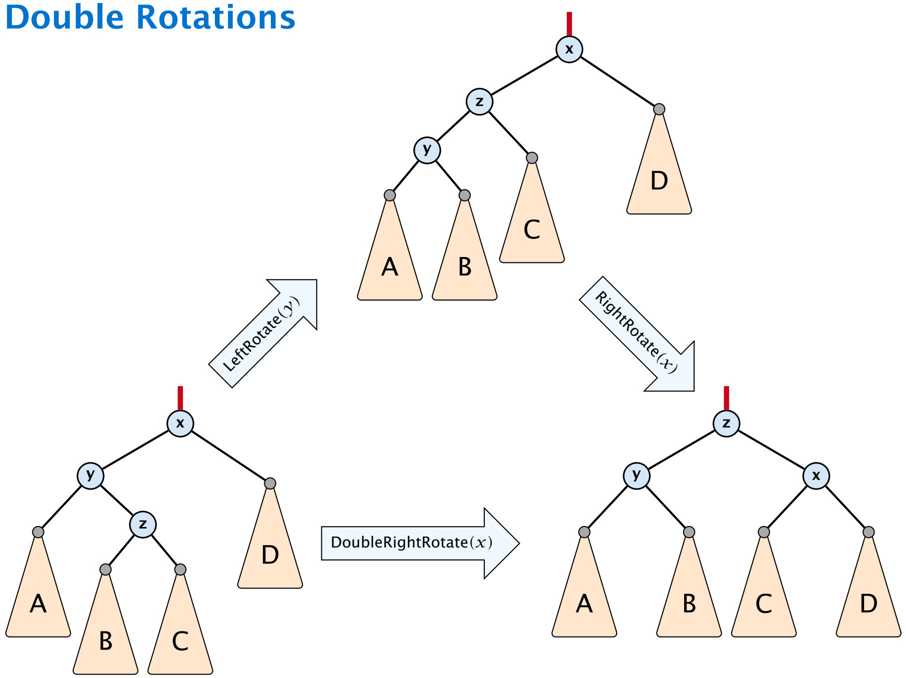
where T_{c_ℓ} and T_{c_r}, are the sub-trees rooted at c_ℓ and c_r, respectively.

Rotations

The properties will be maintained through rotations:



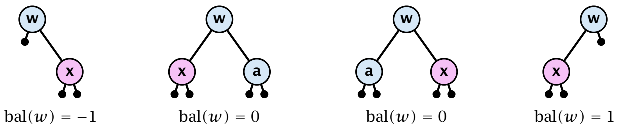
Double Rotations



AVL-trees: Insert

Note that before the insertion w is right above the leaf level, i.e., x replaces a child of w that was a dummy leaf.

- Insert like in a binary search tree.
- Let w denote the parent of the newly inserted node x.
- One of the following cases holds:



- If $\text{bal}[w] \neq 0$, T_w has changed height; the balance-constraint may be violated at ancestors of w.
- Call AVL-fix-up-insert$(\text{parent}[w])$ to restore the balance-condition.

AVL-trees: Insert

Invariant at the beginning of AVL-fix-up-insert(v):

1. The balance constraints hold at all descendants of v.
2. A node has been inserted into T_c, where c is either the right or left child of v.
3. T_c has increased its height by one (otw. we would already have aborted the fix-up procedure).
4. The balance at node c fulfills $\text{balance}[c] \in \{-1, 1\}$. This holds because if the balance of c is 0, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

Note that these constraints hold for the first call AVL-fix-up-insert$(\text{parent}[w])$.

AVL-trees: Insert

Algorithm 11 AVL-fix-up-insert(v)

```
if balance[v] ∈ {−2, 2} then DoRotationInsert(v);
if balance[v] ∈ {0} return;
AVL-fix-up-insert(parent[v]);
```

We will show that the above procedure is correct, and that it will do at most one rotation.

AVL-trees: Insert

Algorithm 12 DoRotationInsert(v)

```
if balance[v] = −2 then // insert in right sub-tree
    if balance[right[v]] = −1 then
        LeftRotate(v);
    else
        DoubleLeftRotate(v);
else // insert in left sub-tree
    if balance[left[v]] = 1 then
        RightRotate(v);
    else
        DoubleRightRotate(v);
```

AVL-trees: Insert

It is clear that the invariant for the fix-up routine holds as long as no rotations have been done.

We have to show that after doing one rotation **all** balance constraints are fulfilled.

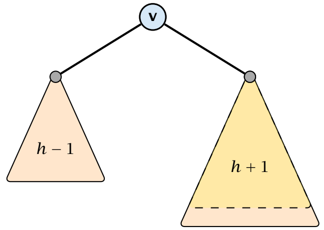
We show that after doing a rotation at v:

- v fulfills balance condition.
- All children of v still fulfill the balance condition.
- The height of T_v is the same as before the insert-operation took place.

We only look at the case where the insert happened into the right sub-tree of v. The other case is symmetric.

AVL-trees: Insert

We have the following situation:

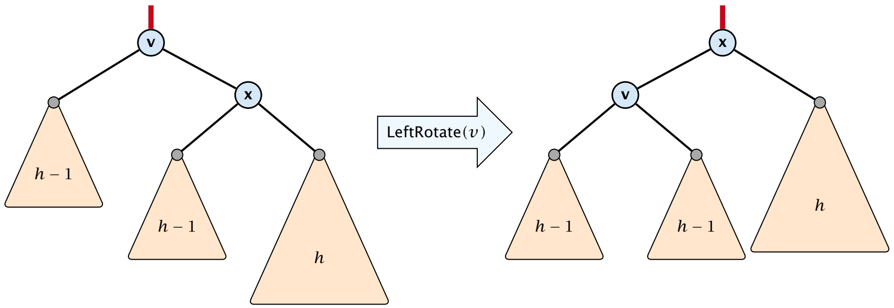


The right sub-tree of v has increased its height which results in a balance of -2 at v.

Before the insertion the height of T_v was $h+1$.

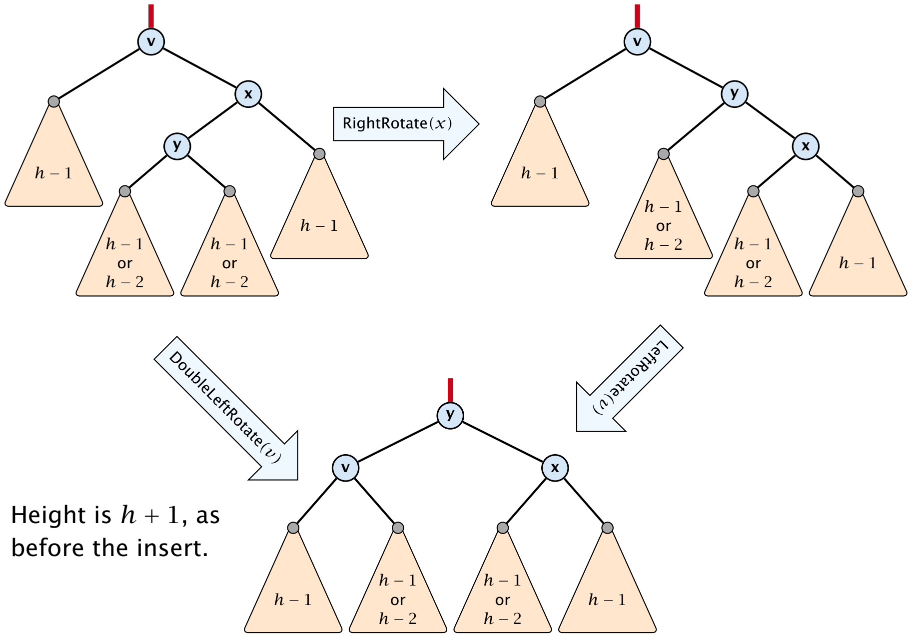
Case 1: balance[right[v]] = -1

We do a left rotation at v



Now, the subtree has height $h+1$ as before the insertion. Hence, we do not need to continue.

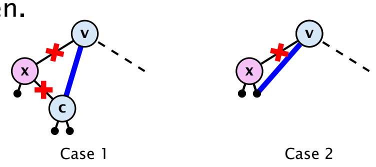
Case 2: balance[right[v]] = 1



Height is $h+1$, as before the insert.

AVL-trees: Delete

- Delete like in a binary search tree.
- Let v denote the parent of the node that has been spliced out.
- The balance-constraint may be violated at v, or at ancestors of v, as a sub-tree of a child of v has reduced its height.
- Initially, the node c—the new root in the sub-tree that has changed—is either a dummy leaf or a node with two dummy leafs as children.



In both cases $\text{bal}[c] = 0$.

- Call AVL-fix-up-delete(v) to restore the balance-condition.

AVL-trees: Delete

Invariant at the beginning AVL-fix-up-delete(v):

1. The balance constraints holds at all descendants of v.
2. A node has been deleted from T_c, where c is either the right or left child of v.
3. T_c has decreased its height by one.
4. The balance at the node c fulfills $\text{balance}[c] = 0$. This holds because if the balance of c is in $\{-1, 1\}$, then T_c did not change its height, and the whole procedure would have been aborted in the previous step.

AVL-trees: Delete

Algorithm 13 AVL-fix-up-delete(v)

```
if balance[v] ∈ {−2, 2} then DoRotationDelete(v);
if balance[v] ∈ {−1, 1} return;
AVL-fix-up-delete(parent[v]);
```

We will show that the above procedure is correct. However, for the case of a delete there may be a logarithmic number of rotations.

AVL-trees: Delete

Algorithm 14 DoRotationDelete(v)

```
if balance[v] = −2 then // deletion in left sub-tree
    if balance[right[v]] ∈ {0, −1} then
        LeftRotate(v);
    else
        DoubleLeftRotate(v);
else // deletion in right sub-tree
    if balance[left[v]] = {0, 1} then
        RightRotate(v);
    else
        DoubleRightRotate(v);
```

Note that the case distinction on the second level ($\text{bal}[\text{right}[v]]$ and $\text{bal}[\text{left}[v]]$) is not done w.r.t. the child c for which the sub-tree T_c has changed. This is different to AVL-fix-up-insert.

AVL-trees: Delete

It is clear that the invariant for the fix-up routine hold as long as no rotations have been done.

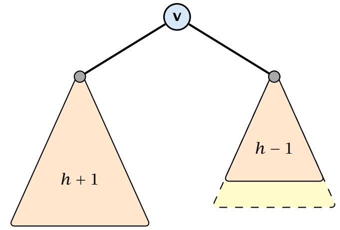
We show that after doing a rotation at v:

- v fulfills the balance condition.
- All children of v still fulfill the balance condition.
- If now $\text{balance}[v] \in \{-1, 1\}$ we can stop as the height of T_v is the same as before the deletion.

We only look at the case where the deleted node was in the right sub-tree of v. The other case is symmetric.

AVL-trees: Delete

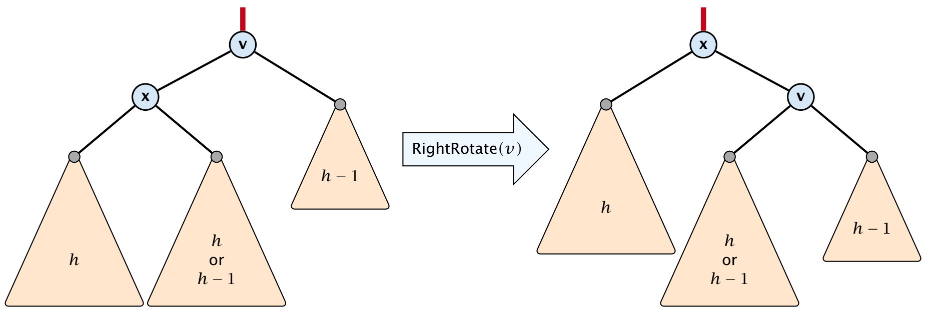
We have the following situation:



The right sub-tree of v has decreased its height which results in a balance of 2 at v.

Before the deletion the height of T_v was $h + 2$.

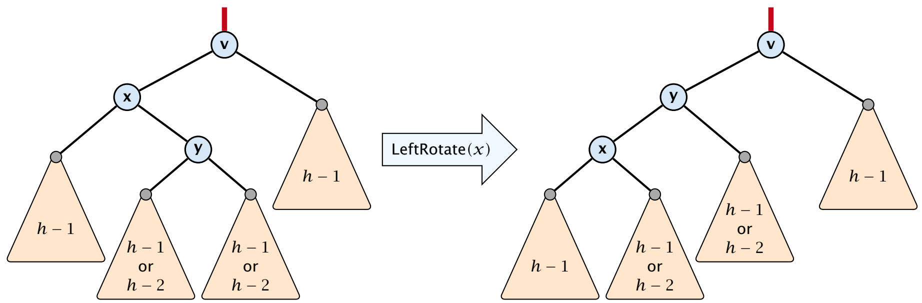
Case 1: $\text{balance}[\text{left}[v]] \in \{0, 1\}$

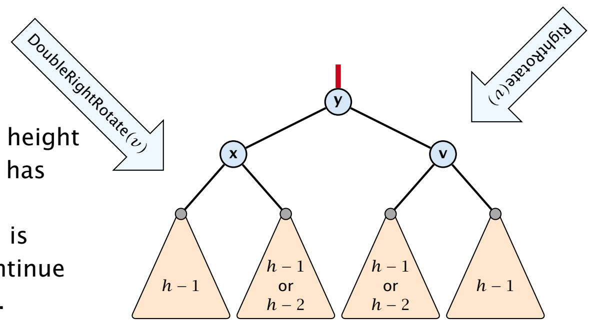


If the middle subtree has height h the whole tree has height $h + 2$ as before the deletion. The iteration stops as the balance at the root is non-zero.

If the middle subtree has height $h - 1$ the whole tree has decreased its height from $h + 2$ to $h + 1$. We do continue the fix-up procedure as the balance at the root is zero.

Case 2: $\text{balance}[\text{left}[v]] = -1$





Sub-tree has height $h + 1$, i.e., it has shrunk. The balance at y is zero. We continue the iteration.