

6 Recurrences

Algorithm 2 mergesort(list L)

```
1:  $n \leftarrow \text{size}(L)$ 
2: if  $n \leq 1$  return  $L$ 
3:  $L_1 \leftarrow L[1 \cdots \lfloor \frac{n}{2} \rfloor]$ 
4:  $L_2 \leftarrow L[\lfloor \frac{n}{2} \rfloor + 1 \cdots n]$ 
5: mergesort( $L_1$ )
6: mergesort( $L_2$ )
7:  $L \leftarrow \text{merge}(L_1, L_2)$ 
8: return  $L$ 
```

This algorithm requires

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathcal{O}(n) \leq 2T\left(\left\lceil \frac{n}{2} \right\rceil\right) + \mathcal{O}(n)$$

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How do we bring the expression for the number of comparisons (\approx running time) into a **closed form**?

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Methods for Solving Recurrences

1. Guessing+Induction

Guess the right solution and prove that it is correct via induction. It needs experience to make the right guess.

2. Master Theorem

For a lot of recurrences that appear in the analysis of algorithms this theorem can be used to obtain tight asymptotic bounds. It does not provide exact solutions.

3. Characteristic Polynomial

Linear homogenous recurrences can be solved via this method.

Methods for Solving Recurrences

4. Generating Functions

A more general technique that allows to solve certain types of linear inhomogenous relations and also sometimes non-linear recurrence relations.

5. Transformation of the Recurrence

Sometimes one can transform the given recurrence relations so that it e.g. becomes linear and can therefore be solved with one of the other techniques.

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First we need to get rid of the \mathcal{O} -notation in our recurrence:

$$T(n) \leq \begin{cases} 2T(\lceil \frac{n}{2} \rceil) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that instead we had

$$T(n) \leq \begin{cases} 2T(\frac{n}{2}) + cn & n \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

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Formally one would make an induction proof, where the above is the induction step. The base case is usually trivial.

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Hence, statement is **true** if we choose $d \geq c$.

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If we do not do this we instead consider the following recurrence:

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Note that we can do this as for constant-sized inputs the running time is always some constant (b in the above case).

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$$\leq dn \log n + (\log 9 - 3.5)dn + cn$$

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$$\leq dn \log n$$

for a suitable choice of d .

6.2 Master Theorem

Lemma 1

Let $a \geq 1$, $b \geq 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

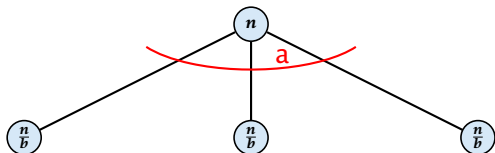
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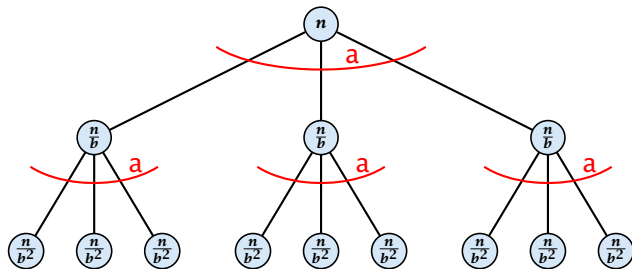
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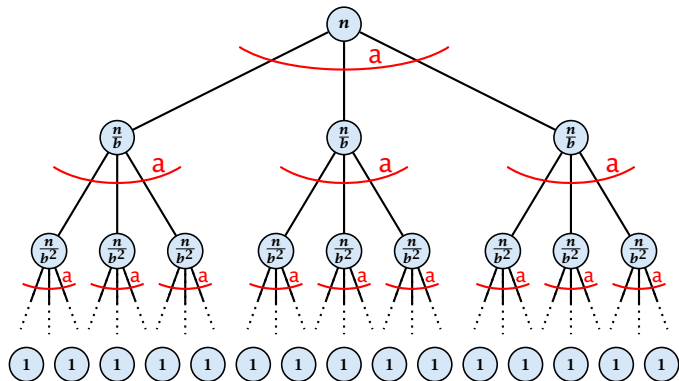
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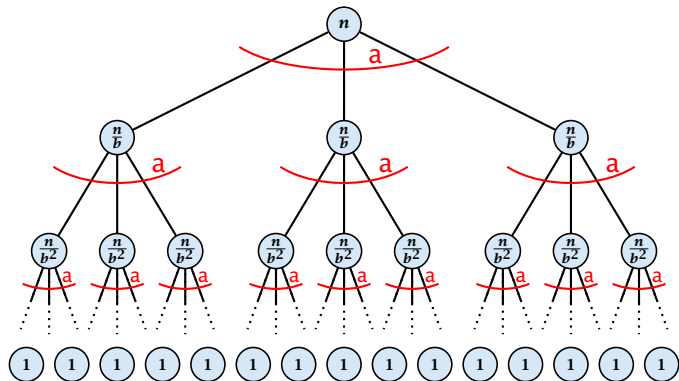
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

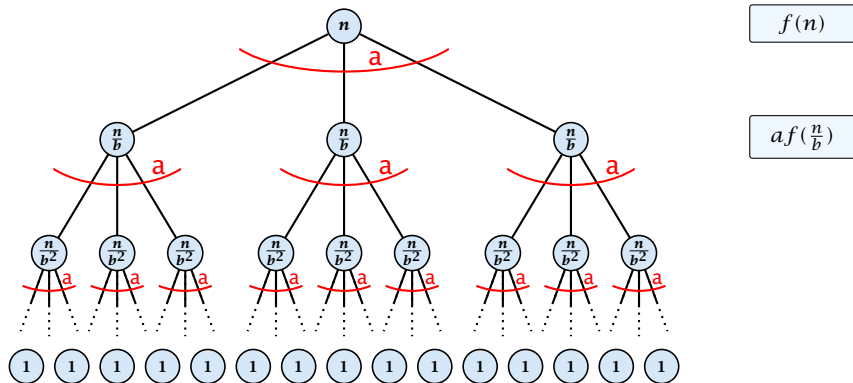
The running time of a recursive algorithm can be visualized by a recursion tree:



$f(n)$

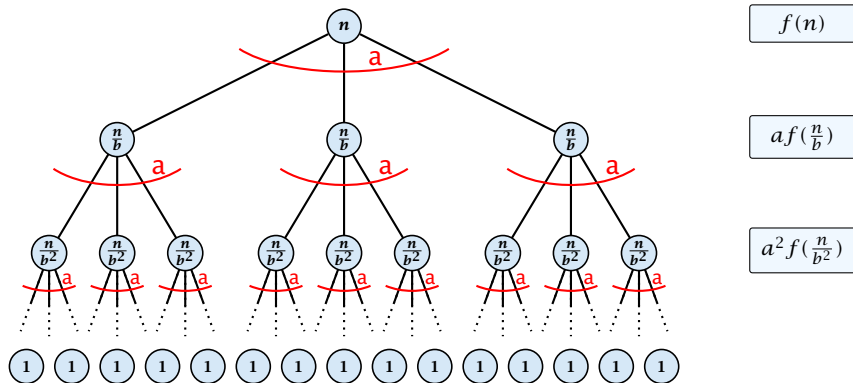
The Recursion Tree

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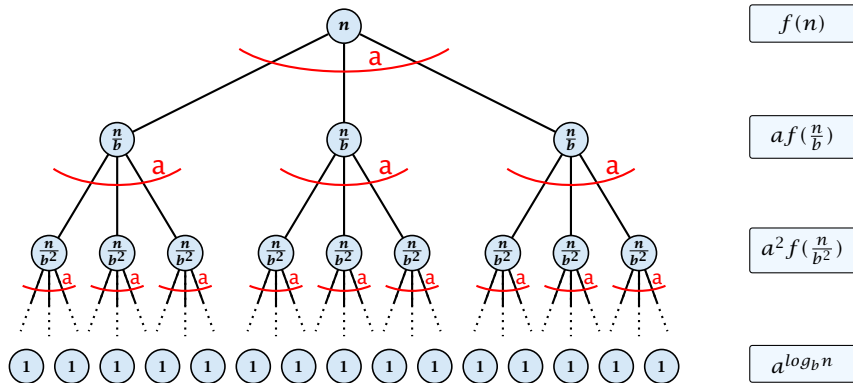
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



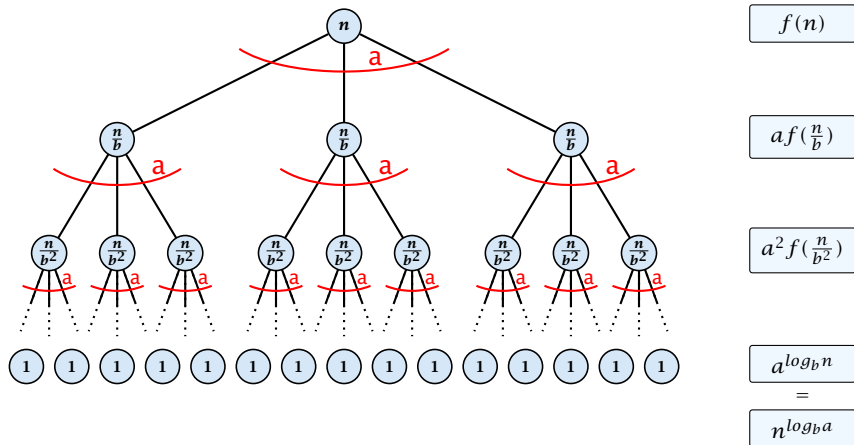
The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:



6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) .$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$T(n) = n^{\log_b a}$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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$$\begin{aligned} \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \end{aligned}$$

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Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

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$$\boxed{b^{-i(\log_b a - \epsilon)} = b^{\epsilon i} (b^{\log_b a})^{-i} = b^{\epsilon i} a^{-i}} = cn^{\log_b a - \epsilon} \sum_{i=0}^{\log_b n - 1} (b^{\epsilon})^i$$

$$\begin{aligned} \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \\ &= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon}) \end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1}\right) n^{\log_b(a)}$$

Case 1. Now suppose that $f(n) \leq cn^{\log_b a - \epsilon}$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\begin{aligned} \boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} &= cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1) \\ &= cn^{\log_b a - \epsilon} (n^{\epsilon} - 1) / (b^{\epsilon} - 1) \\ &= \frac{c}{b^{\epsilon} - 1} n^{\log_b a} (n^{\epsilon} - 1) / (n^{\epsilon}) \end{aligned}$$

Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1}\right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a}$.

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \mathcal{O}(n^{\log_b a} \log_b n) \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log n).$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n)$$

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \quad \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

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$$\boxed{n = b^\ell \Rightarrow \ell = \log_b n} = cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$$

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$$\begin{aligned} \boxed{n = b^\ell \Rightarrow \ell = \log_b n} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \end{aligned}$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

$n = b^\ell \Rightarrow \ell = \log_b n$
--

$$\begin{aligned} &= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k \\ &= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k \\ &= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \end{aligned}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ \leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k$$

$$n = b^\ell \Rightarrow \ell = \log_b n$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b\left(\frac{b^\ell}{b^i}\right)\right)^k$$

$$= cn^{\log_b a} \sum_{i=0}^{\ell-1} (\ell - i)^k$$

$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

Case 2. Now suppose that $f(n) \leq cn^{\log_b a} (\log_b(n))^k$.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a} \cdot \left(\log_b\left(\frac{n}{b^i}\right)\right)^k \end{aligned}$$

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

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From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &= \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &= \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

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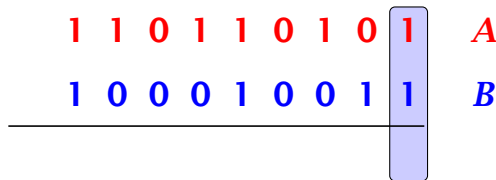
For this we first need to be able to add two integers A and B :

$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline \end{array}$$

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									0

The diagram shows the addition of two 8-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0. The bits of B are 1, 0, 0, 0, 1, 0, 0, 1. A horizontal line is drawn under the bits of B. A vertical bar on the right side of the diagram contains the bits 1, 1, and 0, corresponding to the bits of A, B, and the result of the addition, respectively. A small green '1' is written below the 8th bit of B, indicating a carry-in.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>								1	
									0

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
							1	1	
							0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

The diagram shows the binary addition of two integers, A and B. A horizontal line separates the two numbers. A vertical blue bar highlights the carry propagation from bit 6 to bit 7. The carry is labeled '1' in green below the bar.

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
							1	1	
								0	0

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
						1	1	1	
						0	0	0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical blue box highlights the carry propagation from the 5th bit to the 6th bit. The carry bits are shown as green '1's below the horizontal line. The final result is 000.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
				0	1	1	1		
					1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical bar highlights the carry propagation from the 5th bit to the 6th bit. The result of the addition is shown below the horizontal line, with the carry bits (0, 1, 1, 1) and the final result (1 0 0 0) clearly visible.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
				1	0	1	1	1	
				0	1	0	0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical bar highlights the carry propagation from bit 4 to bit 5. The carry bits are shown in green below the horizontal line.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	0	1	1	1		
			0	1	0	0	0		

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
		1	1	0	1	1	1		
<hr/>			0	0	1	0	0	0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			1	1	0	1	1	1	
			0	0	1	0	0	0	

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
		1	0	0	1	0	0	0	

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1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
	0	1	1	0	1	1	1		
		1	0	0	1	0	0	0	

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	1	1	0	1	1	0	1	0	1	A
	1	0	0	0	1	0	0	1	1	B
	<small>0</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>0</small>	<small>1</small>	<small>1</small>	<small>1</small>		
<hr/>										
	1	1	0	0	1	0	0	0		

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
	0	0	1	1	0	1	1	1	
1	1	0	0	1	0	0	0	0	

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1	0	0	0	1	0	0	1	1		B
<hr/>										
	0	1	1	0	0	1	0	0	0	

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1	0	0	1	1	0	1	1	1		
	0	1	1	0	0	1	0	0	0	

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This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

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$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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Time requirement:

Example: Multiplying Two Integers

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$:
 $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

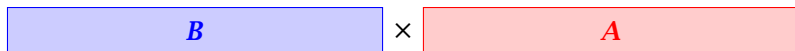
A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

Example: Multiplying Two Integers

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Suppose that integers A and B are of length $n = 2^k$, for some k .


$$\boxed{B} \times \boxed{A}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_n \quad \dots \quad b_0} \times \boxed{a_n \quad \dots \quad a_0}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\begin{array}{ccccccc} b_n & \cdots & b_{\frac{n}{2}} & b_{\frac{n}{2}-1} & \cdots & b_0 & \\ \times & & a_n & \cdots & a_{\frac{n}{2}} & a_{\frac{n}{2}-1} & \cdots & a_0 \end{array}$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\begin{array}{|c|c|} \hline B_1 & B_0 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline A_1 & A_0 \\ \hline \end{array}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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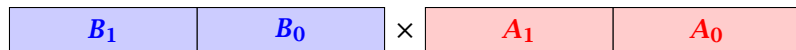
Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 \cdot B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

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```
1: if  $|A| = |B| = 1$  then  
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```

$\mathcal{O}(1)$

Example: Multiplying Two Integers

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$\mathcal{O}(1)$

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$\mathcal{O}(1)$

3: split A into A_0 and A_1

4: split B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$

7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

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$\mathcal{O}(1)$

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$\mathcal{O}(1)$

3: split A into A_0 and A_1

$\mathcal{O}(n)$

4: split B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

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7: $Z_0 \leftarrow \text{mult}(A_0, B_0)$

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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

- 1: **if** $|A| = |B| = 1$ **then** $\mathcal{O}(1)$
- 2: **return** $a_0 \cdot b_0$ $\mathcal{O}(1)$
- 3: split A into A_0 and A_1 $\mathcal{O}(n)$
- 4: split B into B_0 and B_1 $\mathcal{O}(n)$
- 5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$
- 6: $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$
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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

2: **return** $a_0 \cdot b_0$

3: split A into A_0 and A_1

4: split B into B_0 and B_1

5: $Z_2 \leftarrow \text{mult}(A_1, B_1)$

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8: **return** $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$

$\mathcal{O}(1)$

$\mathcal{O}(1)$

$\mathcal{O}(n)$

$\mathcal{O}(n)$

$T\left(\frac{n}{2}\right)$

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$T(\frac{n}{2})$

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Example: Multiplying Two Integers

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2: return $a_0 \cdot b_0$	$\mathcal{O}(1)$
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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

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We can use the following identity to compute Z_1 :

$$\begin{aligned} Z_1 &= A_1 B_0 + A_0 B_1 && = Z_2 && = Z_0 \\ &= (A_0 + A_1) \cdot (B_0 + B_1) - \underbrace{A_1 B_1}_{Z_2} - \underbrace{A_0 B_0}_{Z_0} \end{aligned}$$

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8: return $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$	$\mathcal{O}(n)$

Example: Multiplying Two Integers

We get the following recurrence:

$$T(n) = 3T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Master Theorem: Recurrence: $T[n] = aT\left(\frac{n}{b}\right) + f(n)$.

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

A huge improvement over the “school method”.

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6.3 The Characteristic Polynomial

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \dots + c_kT(n-k) = f(n)$$

This is the general form of a **linear** recurrence relation of **order** k with constant coefficients ($c_0, c_k \neq 0$).

$T(n)$ only depends on the k preceding values. This means the recurrence relation is of **order** k .

The recurrence is **linear** as there are no products of $T(n)$.

When $f(n) = 0$, then the recurrence relation becomes a **linear** recurrence relation of **order** k .

Note that we ignore **boundary conditions** for the moment.

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- ▶ $T(n)$ only depends on the k preceding values. This means the recurrence relation is of **order** k .
- ▶ The recurrence is linear as there are no products of $T[n]$'s.
- ▶ If $f(n) = 0$ then the recurrence relation becomes a linear, **homogenous** recurrence relation of order k .

Note that we ignore **boundary conditions** for the moment.

6.3 The Characteristic Polynomial

Observations:

- ▶ The solution $T[1], T[2], T[3], \dots$ is completely determined by a set of **boundary conditions** that specify values for $T[1], \dots, T[k]$.
- ▶ In fact, any k consecutive values completely determine the solution.
- ▶ k non-consecutive values might not be an appropriate set of boundary conditions (depends on the problem).

Approach:

- ▶ First determine all solutions that satisfy recurrence relation.
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The Homogenous Case

The solution space

$$S = \left\{ \mathcal{T} = T[1], T[2], T[3], \dots \mid \mathcal{T} \text{ fulfills recurrence relation} \right\}$$

is a **vector space**. This means that if $\mathcal{T}_1, \mathcal{T}_2 \in S$, then also $\alpha\mathcal{T}_1 + \beta\mathcal{T}_2 \in S$, for arbitrary constants α, β .

How do we find a non-trivial solution?

We guess that the solution is of the form λ^n , $\lambda \neq 0$, and see what happens. In order for this guess to fulfill the recurrence we need

$$c_0\lambda^n + c_1\lambda^{n-1} + c_2 \cdot \lambda^{n-2} + \dots + c_k \cdot \lambda^{n-k} = 0$$

for all $n \geq k$.

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Dividing by λ^{n-k} gives that all these constraints are identical to

$$c_0\lambda^k + c_1\lambda^{k-1} + c_2 \cdot \lambda^{k-2} + \dots + c_k = 0$$

This means that if λ_i is a root (Nullstelle) of $P[\lambda]$ then $T[n] = \lambda_i^n$ is a solution to the recurrence relation.

Let $\lambda_1, \dots, \lambda_k$ be the k (complex) roots of $P[\lambda]$. Then, because of the vector space property

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Lemma 2

Assume that the characteristic polynomial has k *distinct* roots $\lambda_1, \dots, \lambda_k$. Then *all* solutions to the recurrence relation are of the form

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Proof.

There is one solution for every possible choice of boundary conditions for $T[1], \dots, T[k]$.

We show that the above set of solutions contains one solution for every choice of boundary conditions.

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Proof (cont.).

Suppose I am given boundary conditions $T[i]$ and I want to see whether I can choose the α'_i 's such that these conditions are met:

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We show that the column vectors are linearly independent. Then the above equation has a solution.

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} =$$

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_{k-1}^{k-1} & \lambda_k^{k-1} \end{vmatrix}$$

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 = \prod_{i=1}^k \lambda_i \cdot \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix}$$

Computing the Determinant

$$\begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-2} & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-2} & \lambda_2^{k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-2} & \lambda_k^{k-1} \end{vmatrix} =$$

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$$\begin{vmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & (\lambda_2 - \lambda_1) \cdot 1 & \cdots & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-3} & (\lambda_2 - \lambda_1) \cdot \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & (\lambda_k - \lambda_1) \cdot 1 & \cdots & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-3} & (\lambda_k - \lambda_1) \cdot \lambda_k^{k-2} \end{vmatrix}$$

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$$\prod_{i=2}^k (\lambda_i - \lambda_1) \cdot \begin{vmatrix} 1 & \lambda_2 & \cdots & \lambda_2^{k-3} & \lambda_2^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-3} & \lambda_k^{k-2} \end{vmatrix}$$

Computing the Determinant

Repeating the above steps gives:

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{k-1} & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{k-1}^2 & \lambda_k^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^k & \lambda_2^k & \cdots & \lambda_{k-1}^k & \lambda_k^k \end{vmatrix} = \prod_{i=1}^k \lambda_i \cdot \prod_{i>\ell} (\lambda_i - \lambda_\ell)$$

Hence, if all λ_i 's are different, then the determinant is non-zero.

The Homogeneous Case

What happens if the roots are not all distinct?

Suppose we have a root λ_i with multiplicity (**Vielfachheit**) at least 2. Then not only is λ_i^n a solution to the recurrence but also $n\lambda_i^{n-1}$.

To see this consider the polynomial

$$P[\lambda] \cdot \lambda^{n-k} = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \dots + c_k\lambda^{n-k}$$

Since λ_i is a root we can write this as $Q[\lambda] \cdot (\lambda - \lambda_i)^2$.

Calculating the derivative gives a polynomial that still has root λ_i .

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This means

$$c_0 n \lambda_i^{n-1} + c_1 (n-1) \lambda_i^{n-2} + \dots + c_k (n-k) \lambda_i^{n-k-1} = 0$$

Hence,

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Suppose λ_i has multiplicity j . We know that

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(after taking the derivative; multiplying with λ ; plugging in λ_i)

Doing this again gives

$$c_0 n^2 \lambda_i^n + c_1 (n-1)^2 \lambda_i^{n-1} + \dots + c_k (n-k)^2 \lambda_i^{n-k} = 0$$

We can continue $j-1$ times.

Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

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Hence, $n^\ell \lambda_i^n$ is a solution for $\ell \in 0, \dots, j-1$.

The Homogeneous Case

Lemma 3

Let $P[\lambda]$ denote the characteristic polynomial to the recurrence

$$c_0T[n] + c_1T[n - 1] + \dots + c_kT[n - k] = 0$$

Let λ_i , $i = 1, \dots, m$ be the (complex) roots of $P[\lambda]$ with multiplicities ℓ_i . Then the general solution to the recurrence is given by

$$T[n] = \sum_{i=1}^m \sum_{j=0}^{\ell_i-1} \alpha_{ij} \cdot (n^j \lambda_i^n) .$$

The full proof is omitted. We have only shown that any choice of α_{ij} 's is a solution to the recurrence.

Example: Fibonacci Sequence

$$T[0] = 0$$

$$T[1] = 1$$

$$T[n] = T[n - 1] + T[n - 2] \text{ for } n \geq 2$$

The characteristic polynomial is

$$\lambda^2 - \lambda - 1$$

Finding the roots, gives

$$\lambda_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1}{2} (1 \pm \sqrt{5})$$

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$$\alpha \left(\frac{1 + \sqrt{5}}{2} \right) + \beta \left(\frac{1 - \sqrt{5}}{2} \right) = 1 \implies \alpha - \beta = \frac{2}{\sqrt{5}}$$

Example: Fibonacci Sequence

Hence, the solution is

$$\frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

The Inhomogeneous Case

Consider the recurrence relation:

$$c_0T(n) + c_1T(n-1) + c_2T(n-2) + \cdots + c_kT(n-k) = f(n)$$

with $f(n) \neq 0$.

While we have a fairly general technique for solving **homogeneous**, linear recurrence relations the inhomogeneous case is different.

The Inhomogeneous Case

The general solution of the recurrence relation is

$$T(n) = T_h(n) + T_p(n) ,$$

where T_h is **any** solution to the homogeneous equation, and T_p is **one** particular solution to the inhomogeneous equation.

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The Inhomogeneous Case

Example:

$$T[n] = T[n - 1] + 1 \quad T[0] = 1$$

Then,

$$T[n - 1] = T[n - 2] + 1 \quad (n \geq 2)$$

Subtracting the first from the second equation gives,

$$T[n] - T[n - 1] = T[n - 1] - T[n - 2] \quad (n \geq 2)$$

or

$$T[n] = 2T[n - 1] - T[n - 2] \quad (n \geq 2)$$

I get a completely determined recurrence if I add $T[0] = 1$ and $T[1] = 2$.

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$T[0] = 1$ gives $\alpha = 1$.

$T[1] = 2$ gives $1 + \beta = 2 \Rightarrow \beta = 1$.

The Inhomogeneous Case

If $f(n)$ is a polynomial of degree r this method can be applied $r + 1$ times to obtain a homogeneous equation:

$$T[n] = T[n - 1] + n^2$$

Shift:

$$T[n - 1] = T[n - 2] + (n - 1)^2$$

Difference:

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Shift:

$$T[n - 1] = 2T[n - 2] - T[n - 3] + 2(n - 1) - 1$$

$$T[n] = 2T[n - 1] - T[n - 2] + 2n - 1$$

Shift:

$$\begin{aligned} T[n - 1] &= 2T[n - 2] - T[n - 3] + 2(n - 1) - 1 \\ &= 2T[n - 2] - T[n - 3] + 2n - 3 \end{aligned}$$

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and so on...

6.4 Generating Functions

Definition 4 (Generating Function)

Let $(a_n)_{n \geq 0}$ be a sequence. The corresponding

- ▶ **generating function (Erzeugendenfunktion)** is

$$F(z) := \sum_{n \geq 0} a_n z^n;$$

- ▶ **exponential generating function (exponentielle Erzeugendenfunktion)** is

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6.4 Generating Functions

There are two different views:

A generating function is a **formal power series** (formale Potenzreihe).

Then the generating function is an **algebraic object**.

Let $f = \sum_{n \geq 0} a_n z^n$ and $g = \sum_{n \geq 0} b_n z^n$.

- ▶ **Equality:** f and g are equal if $a_n = b_n$ for all n .
- ▶ **Addition:** $f + g := \sum_{n \geq 0} (a_n + b_n) z^n$.
- ▶ **Multiplication:** $f \cdot g := \sum_{n \geq 0} c_n z^n$ with $c = \sum_{p=0}^n a_p b_{n-p}$.

There are no convergence issues here.

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6.4 Generating Functions

The arithmetic view:

We view a power series as a function $f : \mathbb{C} \rightarrow \mathbb{C}$.

Then, it is important to think about convergence/convergence radius etc.

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It means that the power series $1 - z$ and the power series $\sum_{n \geq 0} z^n$ are invers, i.e.,

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

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We know

$$\sum_{n \geq 0} y^n = \frac{1}{1-y}$$

Hence,

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6. The coefficients of the resulting power series are the a_n .

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which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.

6.5 Transformation of the Recurrence

Example 6

$$f_0 = 1$$

$$f_1 = 2$$

$$f_n = f_{n-1} \cdot f_{n-2} \text{ for } n \geq 2 .$$

6.5 Transformation of the Recurrence

Example 6

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$$g_n := \log f_n .$$

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Let $n = 2^k$:

$$g_k = 3^{k+1} - 2^{k+1}, \text{ hence}$$

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