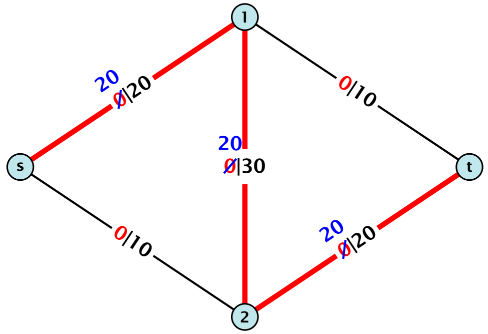
12

Greedy-algorithm:

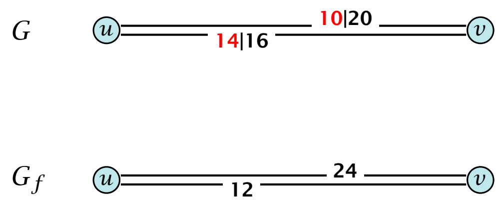
- start with $f(e) = 0$ everywhere
- find an s-t path with $f(e) < c(e)$ on every edge
- augment flow along the path
- repeat as long as possible



The Residual Graph

From the graph $G = (V, E, c)$ and the current flow f we construct an auxiliary graph $G_f = (V, E_f, c_f)$ (the residual graph):

- Suppose the original graph has edges $e_1 = (u, v)$, and $e_2 = (v, u)$ between u and v.
- G_f has edge e'_1 with capacity $\max\{0, c(e_1) - f(e_1) + f(e_2)\}$ and e'_2 with with capacity $\max\{0, c(e_2) - f(e_2) + f(e_1)\}$.



Augmenting Path Algorithm

Definition 50

An augmenting path with respect to flow f, is a path in the auxiliary graph G_f that contains only edges with non-zero capacity.

Algorithm 45 FordFulkerson$(G = (V, E, c))$

1: Initialize $f(e) \leftarrow 0$ for all edges.
2: **while** $\exists$ augmenting path p in G_f **do**
3: augment as much flow along p as possible.

Augmenting Path Algorithm

Theorem 51

*A flow f is a maximum flow **iff** there are no augmenting paths.*

Theorem 52

The value of a maximum flow is equal to the value of a minimum cut.

Proof.

Let f be a flow. The following are equivalent:

1. There exists a cut A, B such that $\text{val}(f) = \text{cap}(A, B)$.
2. Flow f is a maximum flow.
3. There is no augmenting path w.r.t. f.

□

Augmenting Path Algorithm

1. $\Longrightarrow$ 2.
This we already showed.

2. $\Longrightarrow$ 3.
If there were an augmenting path, we could improve the flow. Contradiction.

3. $\Longrightarrow$ 1.

- Let f be a flow with no augmenting paths.
- Let A be the set of vertices reachable from s in the residual graph along non-zero capacity edges.
- Since there is no augmenting path we have $s \in A$ and $t \notin A$.

Augmenting Path Algorithm

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e \in \mathrm{out}(A)} f(e) - \sum_{e \in \mathrm{into}(A)} f(e) \\ &= \sum_{e \in \mathrm{out}(A)} c(e) \\ &= \mathrm{cap}(A, V \setminus A)\end{aligned}$$

This finishes the proof.

Here the first equality uses the flow value lemma, and the second exploits the fact that the flow along incoming edges must be 0 as the residual graph does not have edges leaving A.

Analysis

Assumption:
All capacities are integers between 1 and C.

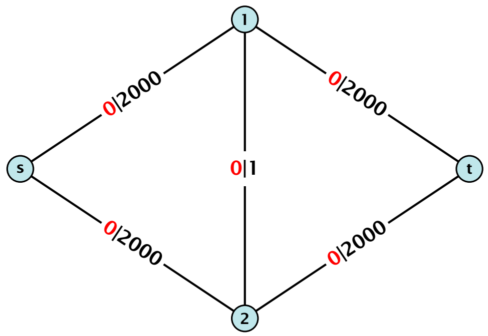
Invariant:
Every flow value $f(e)$ and every residual capacity $c_f(e)$ remains integral troughout the algorithm.

Lemma 53
The algorithm terminates in at most $\mathrm{val}(f^*) \le nC$ iterations, where f^* denotes the maximum flow. Each iteration can be implemented in time $\mathcal{O}(m)$. This gives a total running time of $\mathcal{O}(nmC)$.

Theorem 54
If all capacities are integers, then there exists a maximum flow for which every flow value $f(e)$ is integral.

A bad input

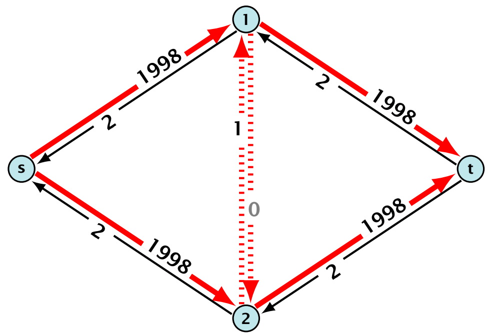
Problem: The running time may not be polynomial.



Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

A bad input

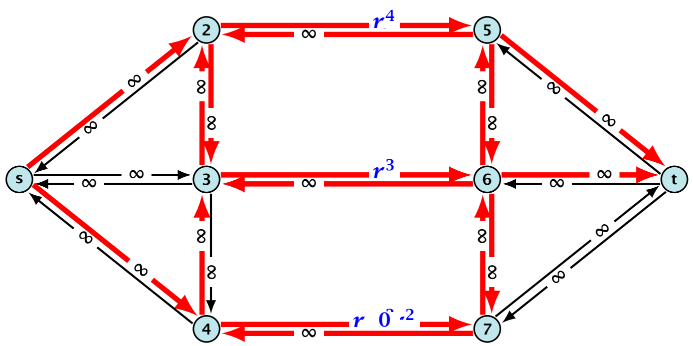
Problem: The running time may not be polynomial.



Question:
Can we tweak the algorithm so that the running time is polynomial in the input length?

A Pathological Input

Let $r = \frac{1}{2}(\sqrt{5} - 1)$. Then $r^{n+2} = r^n - r^{n+1}$.



Running time may be infinite!!!

How to choose augmenting paths?

- We need to find paths efficiently.
- We want to guarantee a small number of iterations.

Several possibilities:

- Choose path with maximum bottleneck capacity.
- Choose path with sufficiently large bottleneck capacity.
- Choose the shortest augmenting path.