# 6.4 Generating Functions

## Definition 7 (Generating Function)

Let $(a_n)_{n\geq 0}$ be a sequence. The corresponding

- generating function (Erzeugendenfunktion) is

$$F(z) := \sum_{n=0}^{\infty} a_n z^n ;$$

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There are two different views:

A generating function is a formal power series (formale Potenzreihe).

Then the generating function is an algebraic object.

Let $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$.

- **Equality:** $f$ and $g$ are equal if $a_n = b_n$ for all $n$.
- **Addition:** $f + g := \sum_{n=0}^{\infty} (a_n + b_n) z^n$.
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We view a power series as a function $f : \mathbb{C} \to \mathbb{C}$.

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What does $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ mean in the algebraic view?

It means that the power series $1 - z$ and the power series $\sum_{n=0}^{\infty} z^n$ are invers, i.e.,

$$\left(1 - z\right) \cdot \left( \sum_{n=0}^{\infty} z^n \right) = 1 \ .$$

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Hence, the generating function of the sequence $a_n = n + 1$ is $1/(1-z)^2$.

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Computing the $k$-th derivative of $\sum z^n$.

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The generating function of the sequence $a_n = \binom{n+k}{k}$ is $\frac{1}{(1-z)^{k+1}}$.

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The generating function of the sequence $a_n = n$ is $\frac{z}{(1-z)^2}$.

# 6.4 Generating Functions

We know

$$\sum_{n\geq 0} y^n = \frac{1}{1-y}$$

Hence,

$$\sum_{n\geq 0} a^n z^n = \frac{1}{1-az}$$

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&= zA(z) + \sum_{n\geq 0} z^n \\
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Hence, $a_n = n + 1$.

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| $n$-th sequence element | generating function |
|---|---|
| $1$ | $\frac{1}{1-z}$ |
| $n+1$ | $\frac{1}{(1-z)^2}$ |
| $\binom{n+k}{n}$ | $\frac{1}{(1-z)^{k+1}}$ |
| $n$ | $\frac{z}{(1-z)^2}$ |
| $a^n$ | $\frac{1}{1-az}$ |
| $n^2$ | $\frac{z(1+z)}{(1-z)^3}$ |
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| $f_{n-k} \quad (n \geq k);\quad 0$ otw. | $z^k F$ |
| $\sum_{i=0}^{n} f_i$ | $\dfrac{F(z)}{1-z}$ |
| $nf_n$ | $z\dfrac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
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| $\sum_{i=0}^{n} f_i$ | $\frac{F(z)}{1-z}$ |
| $nf_n$ | $z\frac{\mathrm{d}F(z)}{\mathrm{d}z}$ |
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6. The coefficients of the resulting power series are the $a_n$.

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&= 1 + 3z \sum_{n\geq 0} a_n z^n + \sum_{n\geq 0} n z^n \\
&= 1 + 3zA(z) + \frac{z}{(1-z)^2}
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$$\begin{aligned} A + B + C &= 1 \\ 2A + 4B + 3C &= 1 \\ A + 3B &= 1 \end{aligned}$$

   which gives

$$A = \frac{7}{4} \quad B = -\frac{1}{4} \quad C = -\frac{1}{2}$$

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&= \frac{7}{4} \cdot \sum_{n \geq 0} 3^n z^n - \frac{1}{4} \cdot \sum_{n \geq 0} z^n - \frac{1}{2} \cdot \sum_{n \geq 0} (n+1) z^n \\
&= \sum_{n \geq 0} \Big( \frac{7}{4} \cdot 3^n - \frac{1}{4} - \frac{1}{2}(n+1) \Big) z^n
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6. This means $a_n = \frac{7}{4}3^n - \frac{1}{2}n - \frac{3}{4}$.