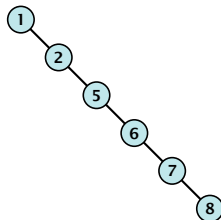
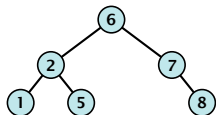


7.1 Binary Search Trees

An (**internal**) **binary search tree** stores the elements in a binary tree. Each tree-node corresponds to an element. All elements in the left sub-tree of a node v have a smaller key-value than $\text{key}[v]$ and elements in the right sub-tree have a larger-key value. We assume that all key-values are different.

(**External** Search Trees store objects only at leaf-vertices)

Examples:

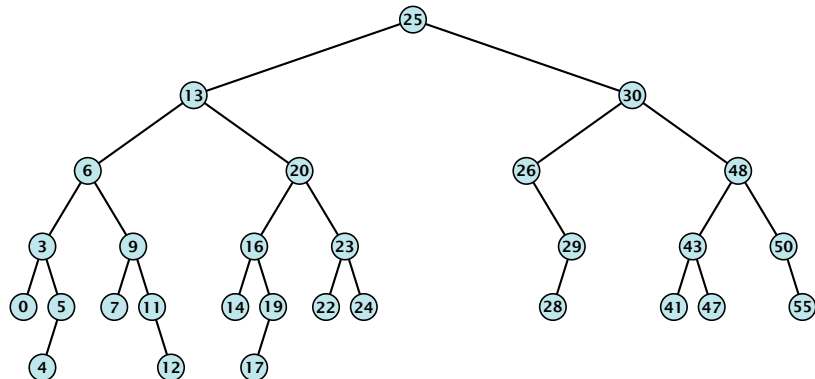


7.1 Binary Search Trees

We consider the following operations on binary search trees. Note that this is a super-set of the dictionary-operations.

- ▶ $T.\text{insert}(x)$
- ▶ $T.\text{delete}(x)$
- ▶ $T.\text{search}(k)$
- ▶ $T.\text{successor}(x)$
- ▶ $T.\text{predecessor}(x)$
- ▶ $T.\text{minimum}()$
- ▶ $T.\text{maximum}()$

Binary Search Trees: Searching

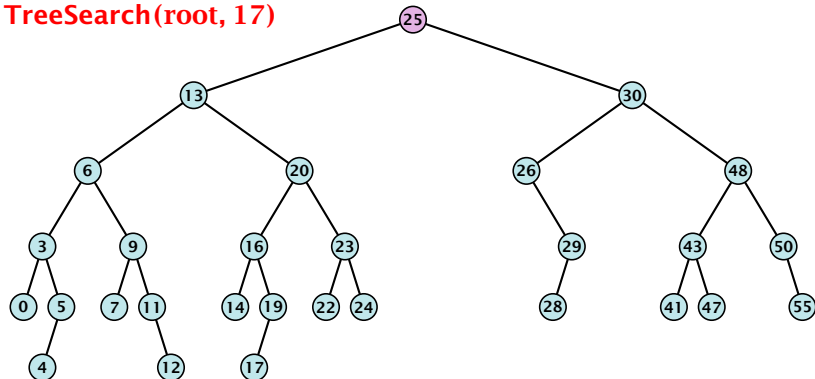


Algorithm 5 $\text{TreeSearch}(x, k)$

- 1: **if** $x = \text{null}$ **or** $k = \text{key}[x]$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** $\text{TreeSearch}(\text{left}[x], k)$
- 3: **else return** $\text{TreeSearch}(\text{right}[x], k)$

Binary Search Trees: Searching

TreeSearch(root, 17)

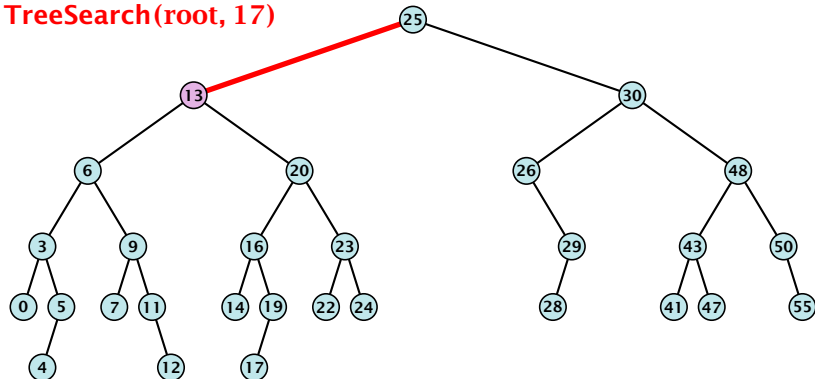


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Binary Search Trees: Searching

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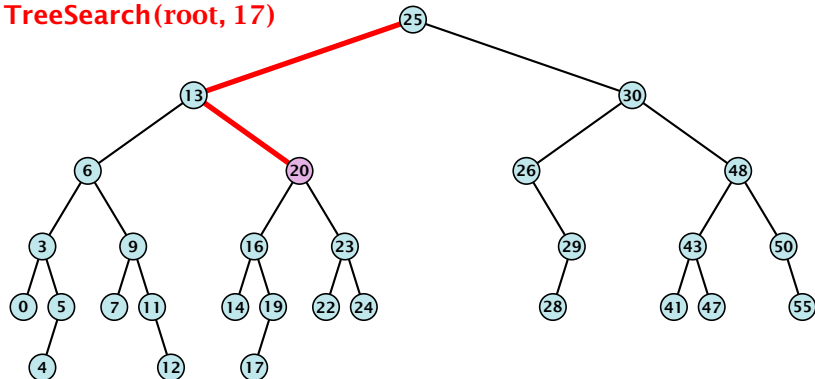


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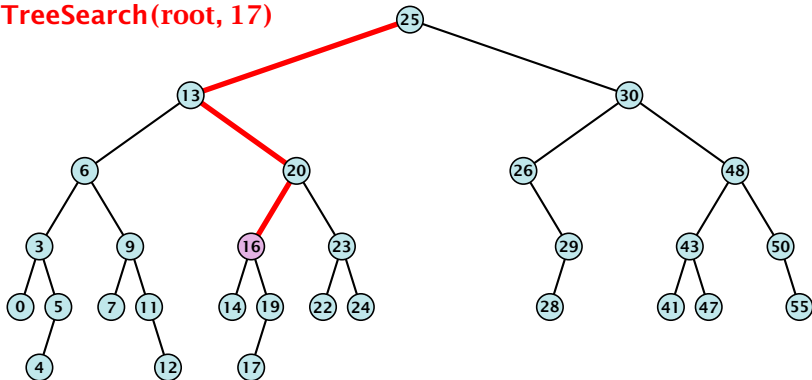


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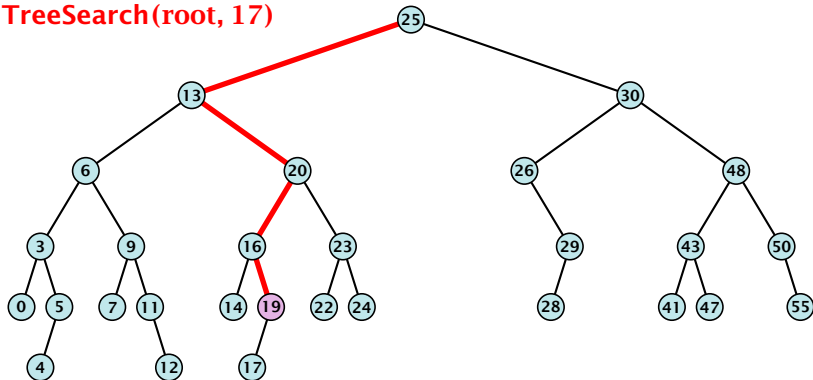


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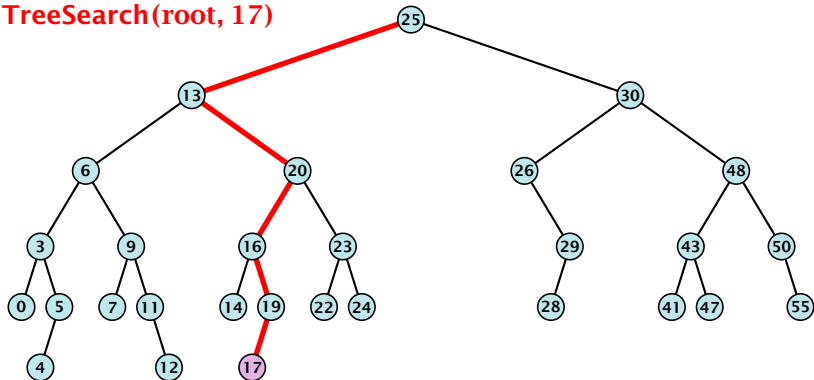


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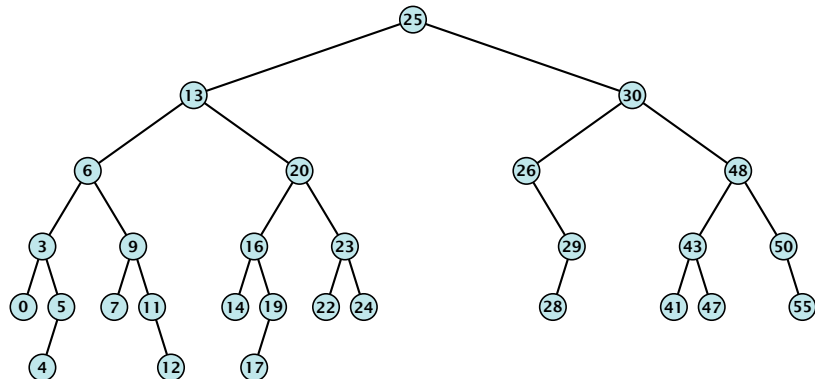
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Binary Search Trees: Searching

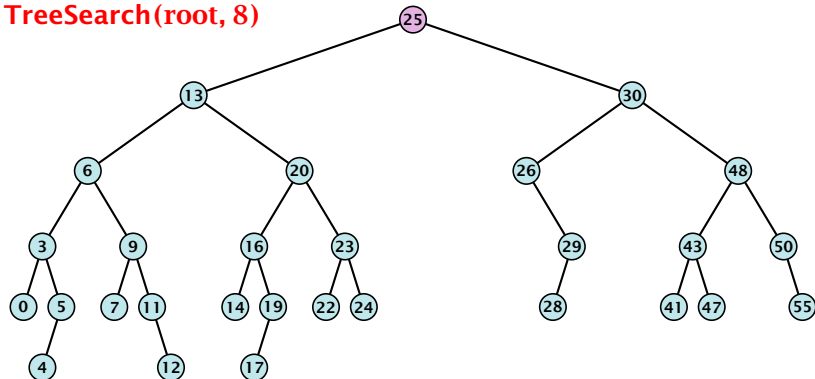


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Binary Search Trees: Searching

TreeSearch(root, 8)

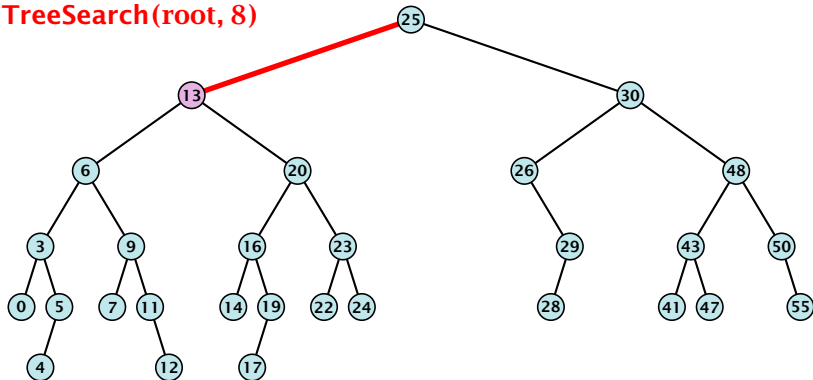


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Binary Search Trees: Searching

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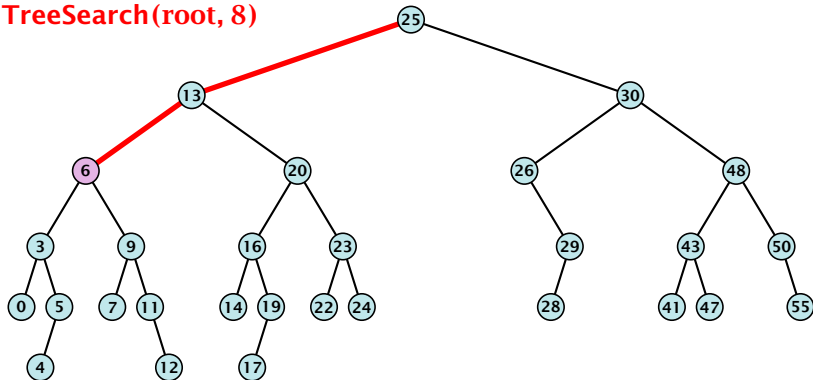


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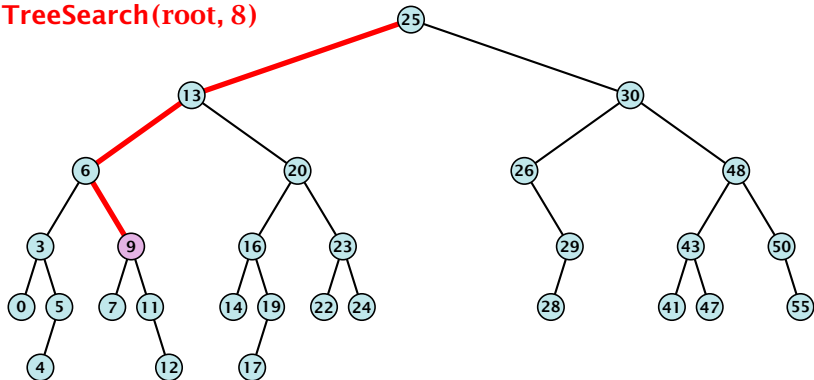


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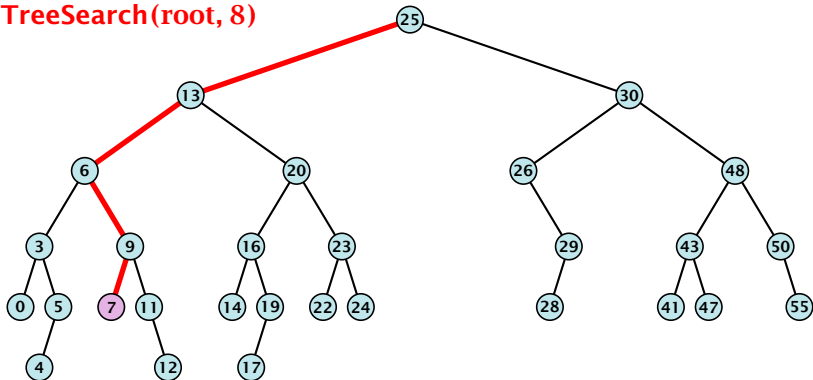


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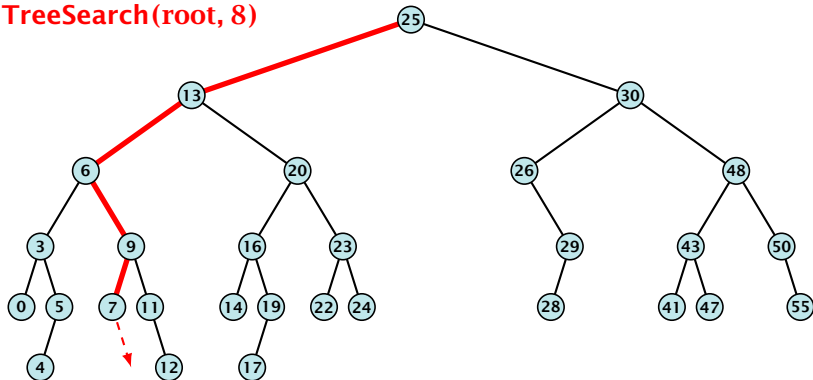


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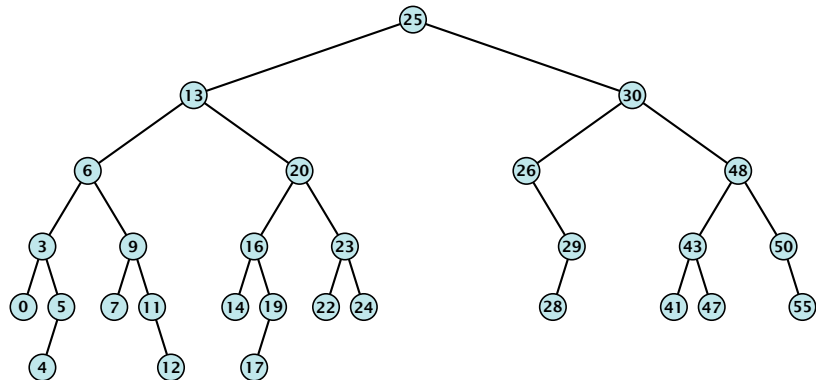
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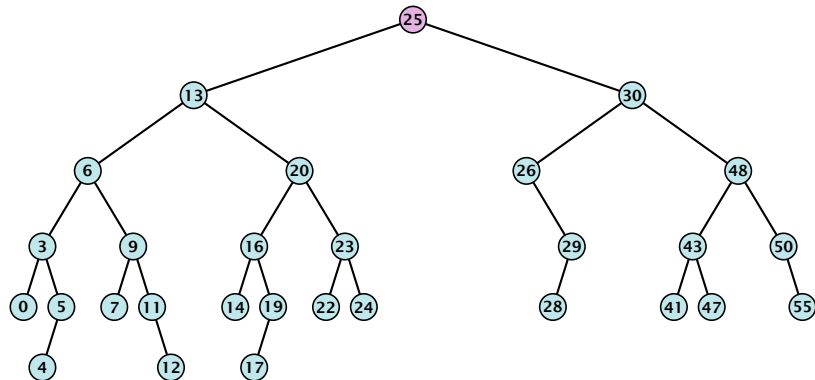
Binary Search Trees: Minimum



Algorithm 6 TreeMin(x)

- 1: **if** $x = \text{null}$ **or** $\text{left}[x] = \text{null}$ **return** x
- 2: **if** $k < \text{key}[x]$ **return** $\text{TreeSearch}(\text{left}[x], k)$
- 3: **else return** $\text{TreeMin}(\text{left}[x])$

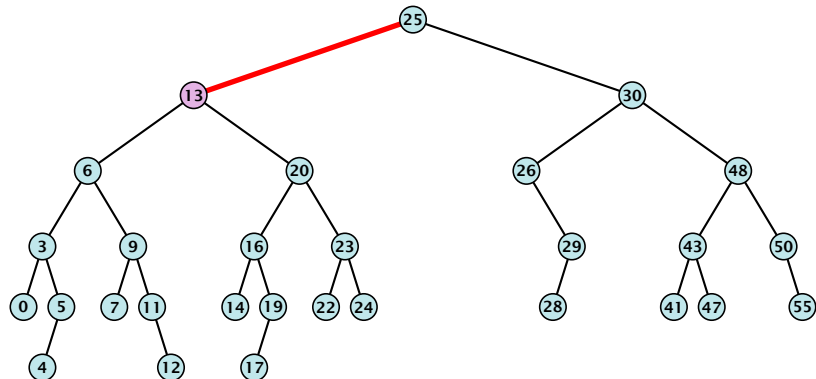
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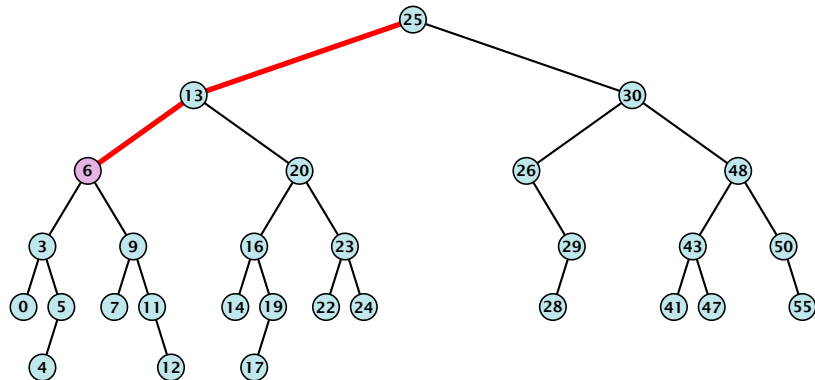
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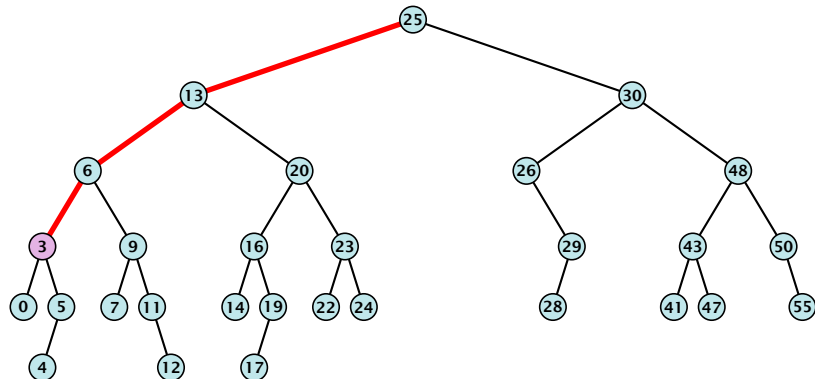
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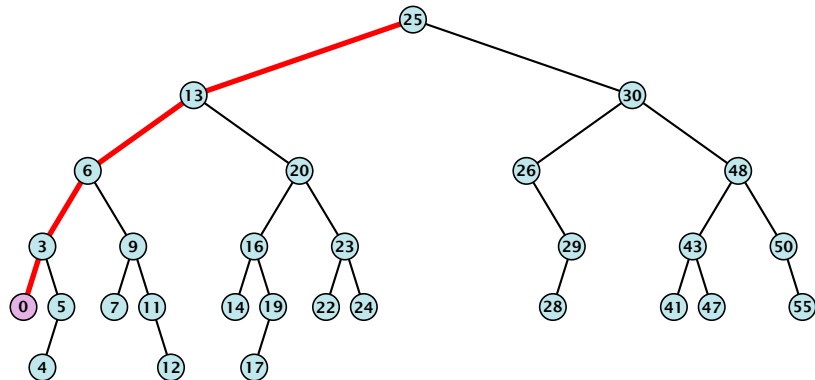
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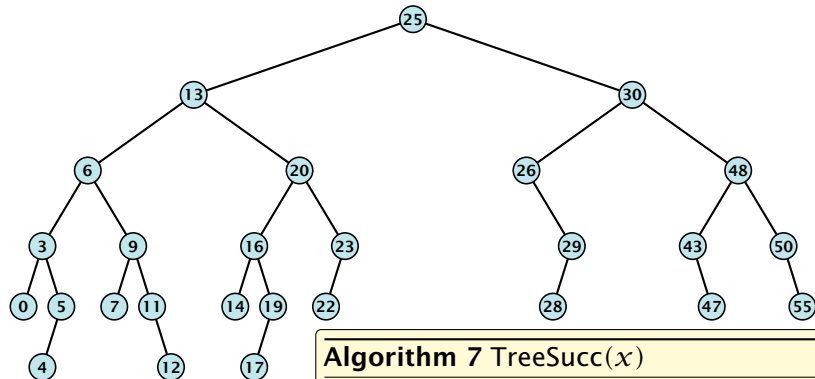
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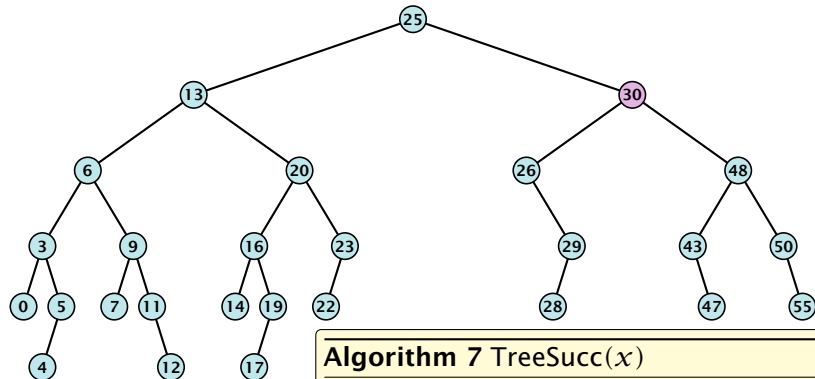
Binary Search Trees: Successor



Algorithm 7 TreeSucc(x)

- 1: **if** right[x] \neq null **return** TreeMin(right[x])
- 2: $y \leftarrow$ parent[x]
- 3: **while** $y \neq$ null **and** $x =$ right[y] **do**
- 4: $x \leftarrow y$; $y \leftarrow$ parent[x]
- 5: **return** y ;

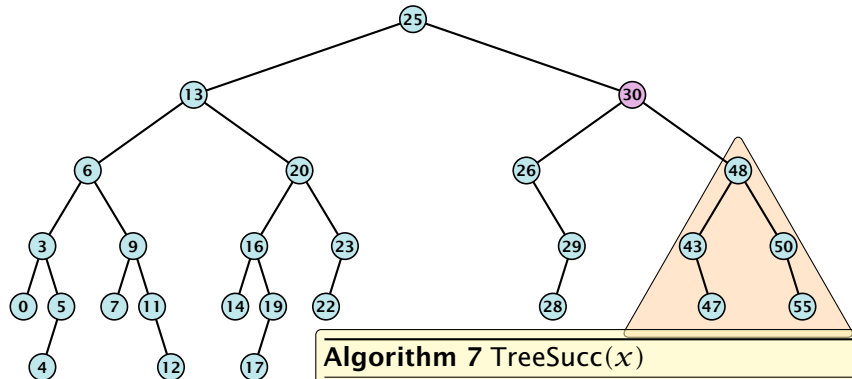
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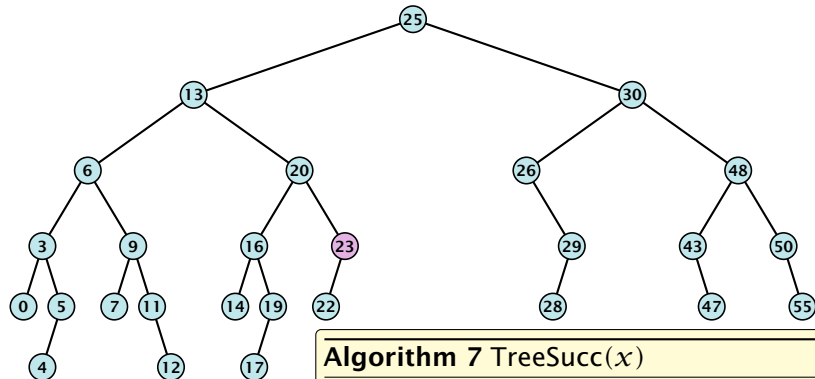
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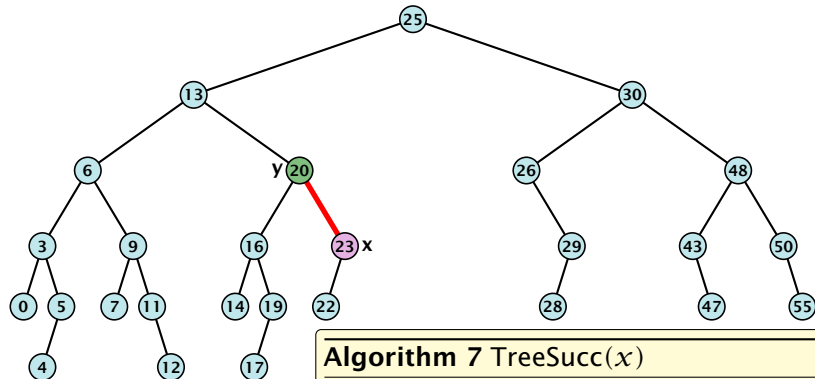
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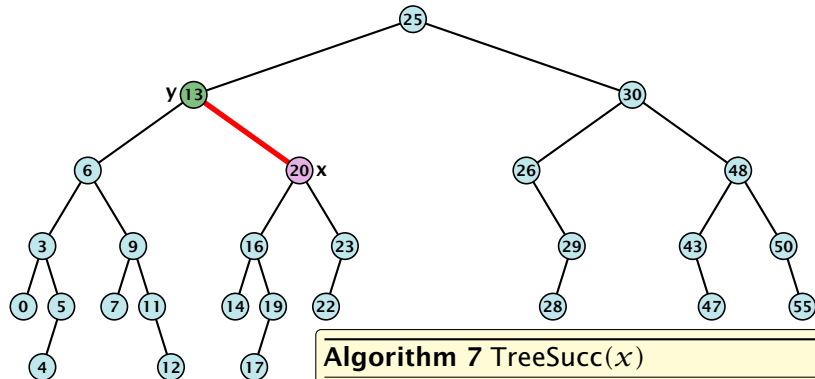
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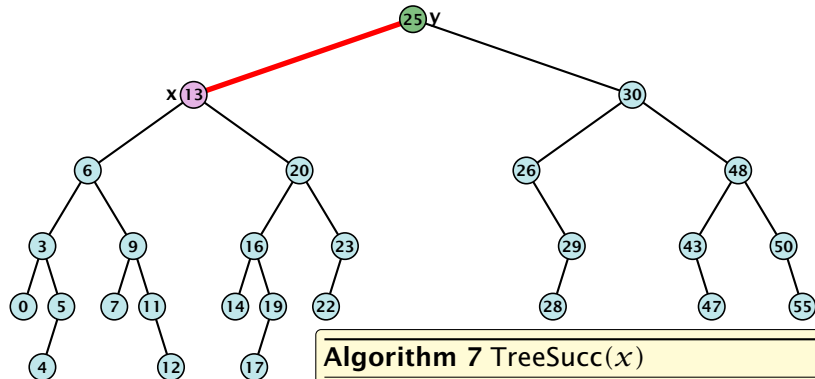
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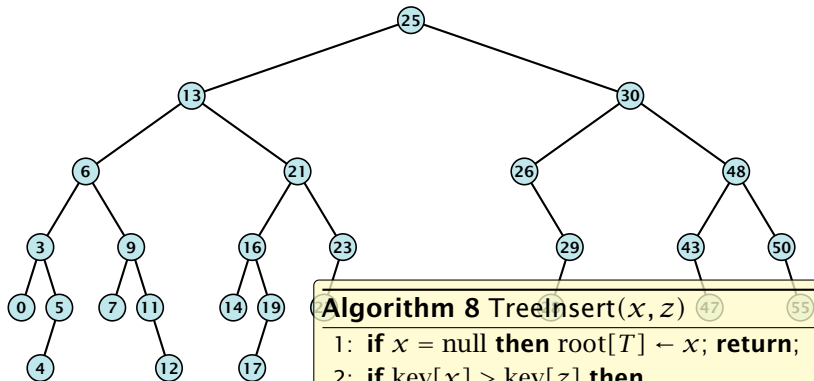
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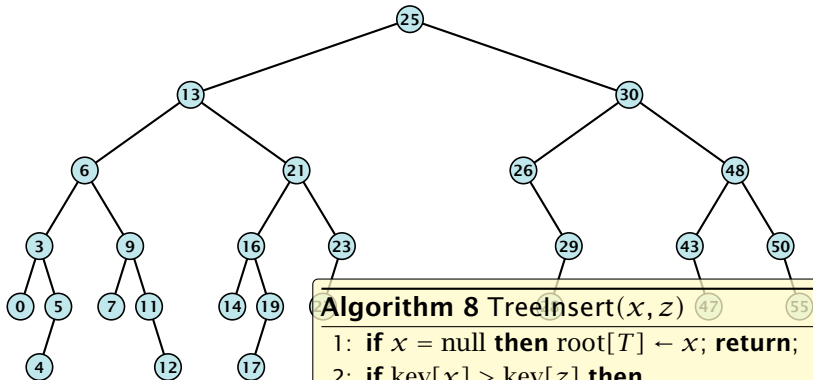


Algorithm 8 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then** $\text{root}[T] \leftarrow x$; **return**;
- 2: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 3: **if** $\text{left}[x] = \text{null}$ **then** $\text{left}[x] \leftarrow z$;
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Binary Search Trees: Insert

Insert element **not** in the tree.

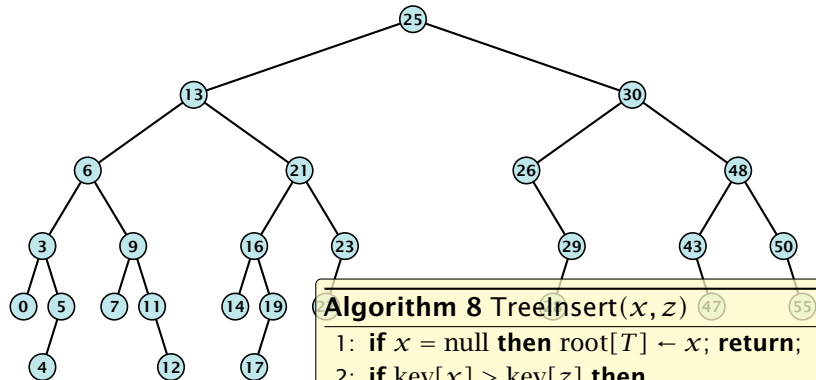


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Search for z . At some point the search stops at a null-pointer. This is the place to insert z .

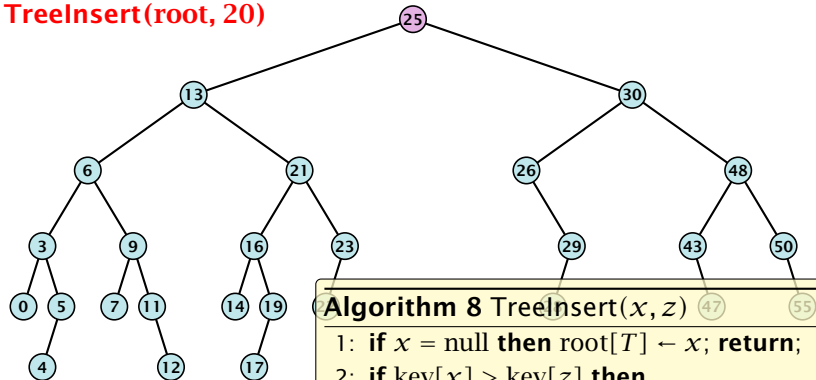
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TreeInsert(root, 20)



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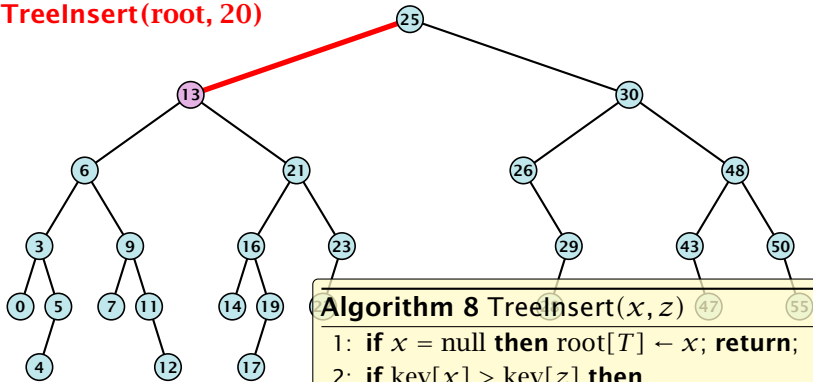
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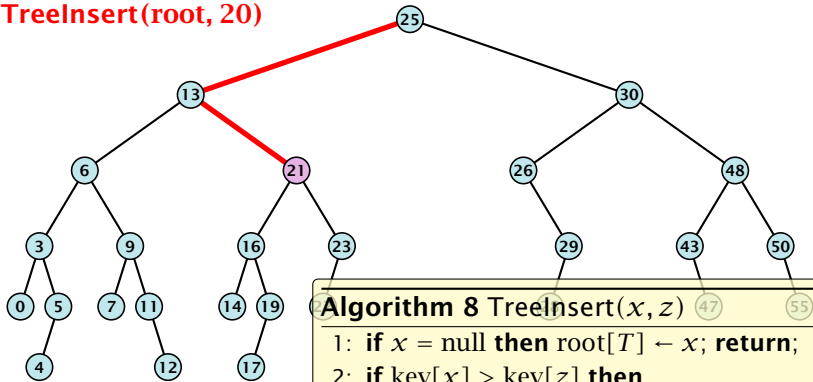
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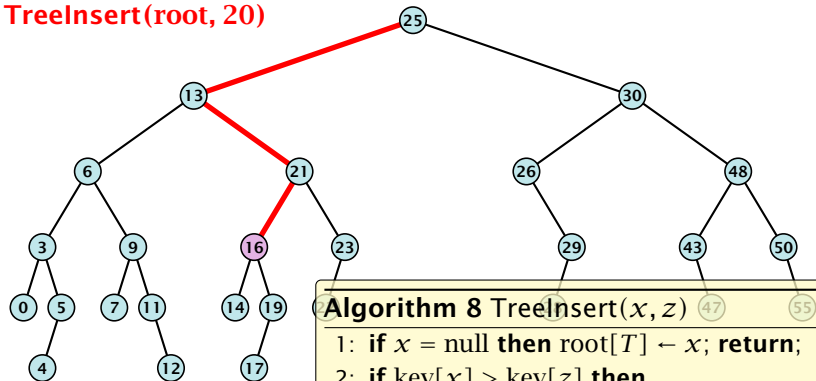
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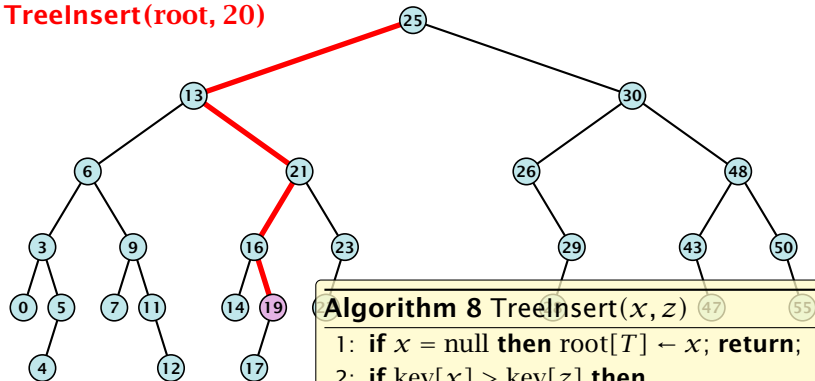
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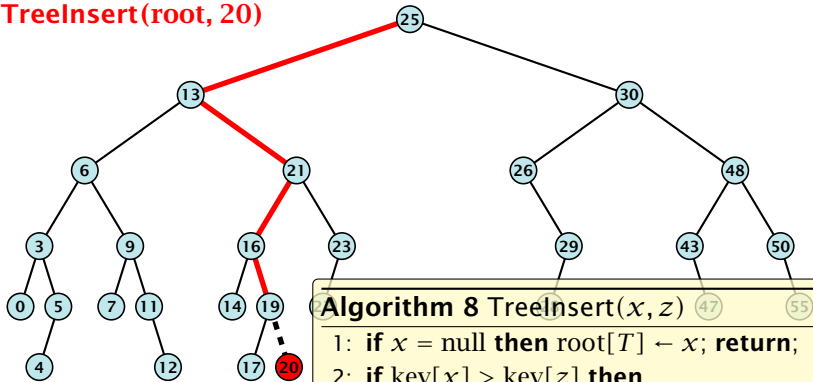
Algorithm 8 TreeInsert(x, z)

- 1: **if** $x = \text{null}$ **then** $\text{root}[T] \leftarrow x$; **return**;
- 2: **if** $\text{key}[x] > \text{key}[z]$ **then**
- 3: **if** $\text{left}[x] = \text{null}$ **then** $\text{left}[x] \leftarrow z$;
- 4: **else** TreeInsert($\text{left}[x], z$);
- 5: **else**
- 6: **if** $\text{right}[x] = \text{null}$ **then** $\text{right}[x] \leftarrow z$;
- 7: **else** TreeInsert($\text{right}[x], z$);
- 8: **return**

Binary Search Trees: Insert

Insert element **not** in the tree.

TreeInsert(root, 20)

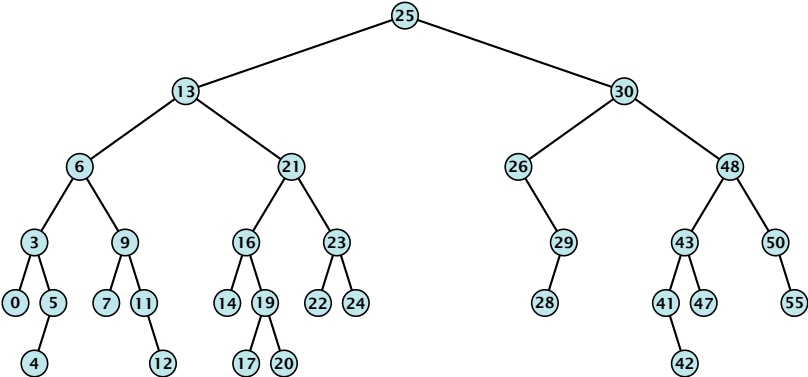


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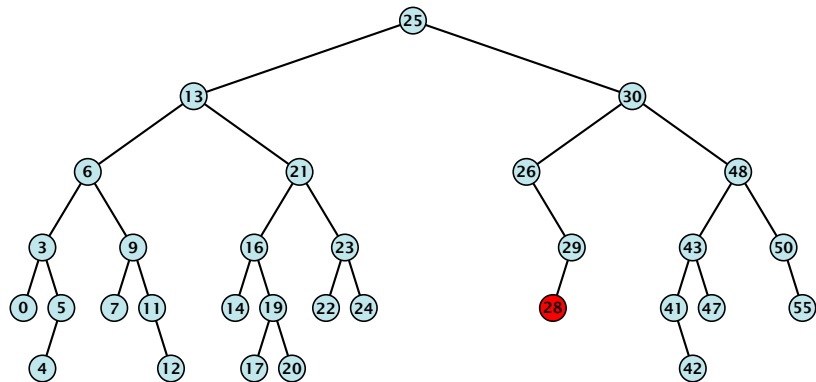
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Binary Search Trees: Delete



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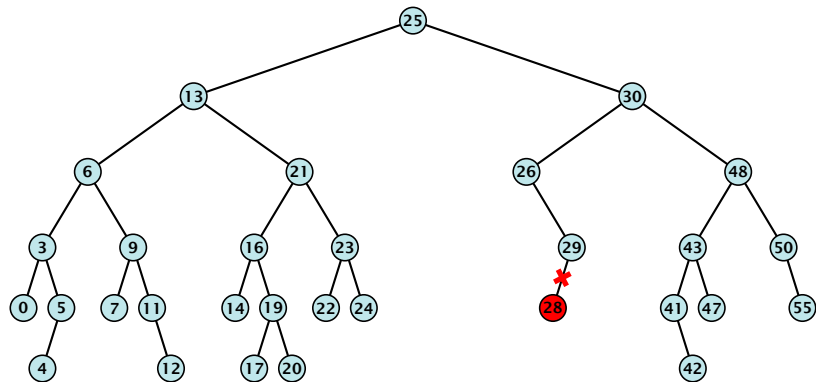


Case 1:

Element does not have any children

- ▶ Simply go to the parent and set the corresponding pointer to null.

Binary Search Trees: Delete

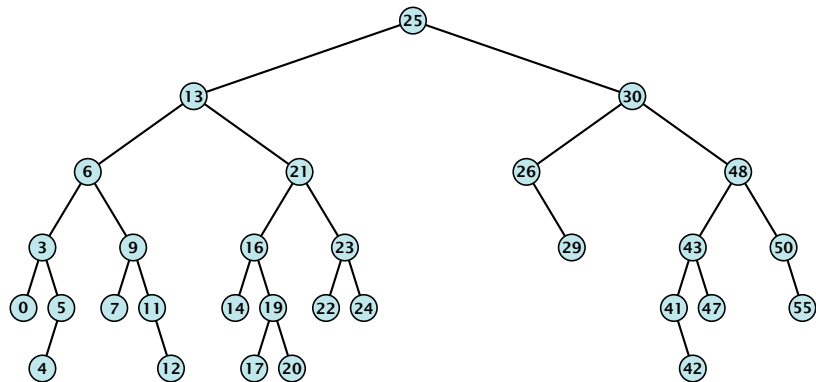


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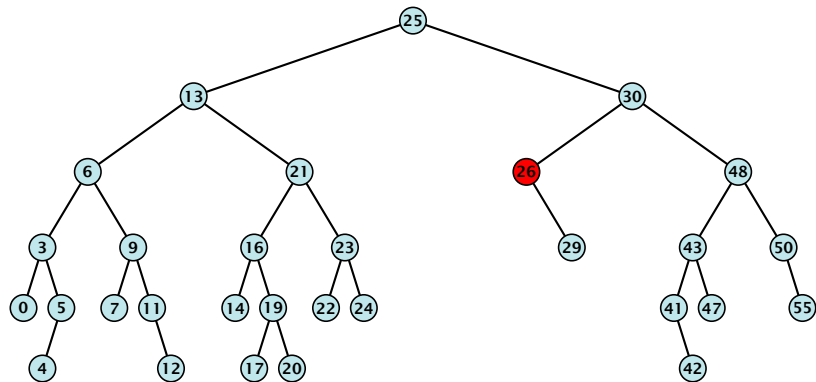


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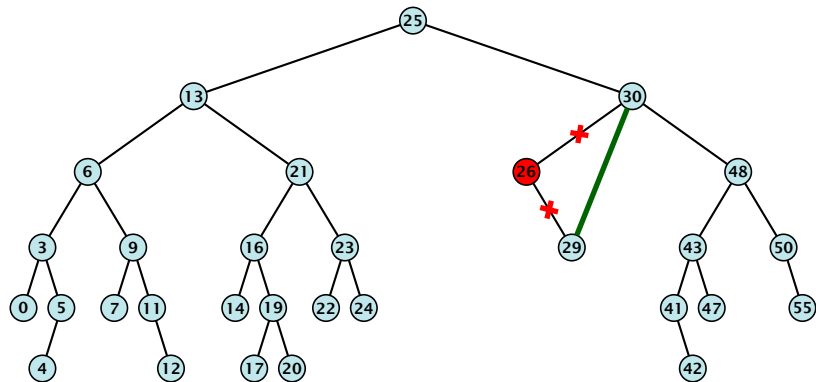


Case 2:

Element has exactly one child

- ▶ Splice the element out of the tree by connecting its parent to its successor.

Binary Search Trees: Delete

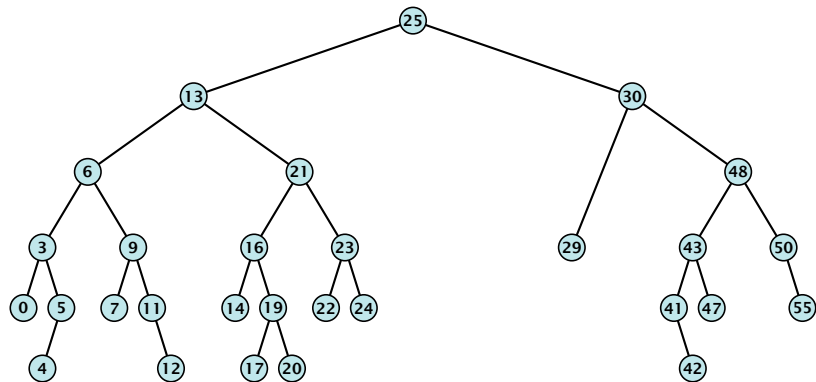


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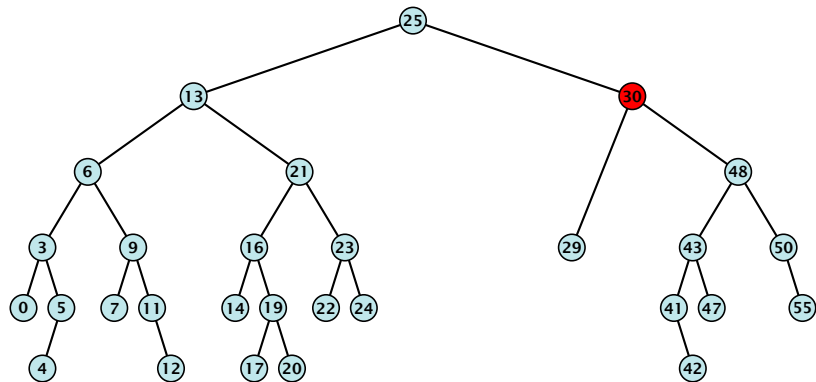


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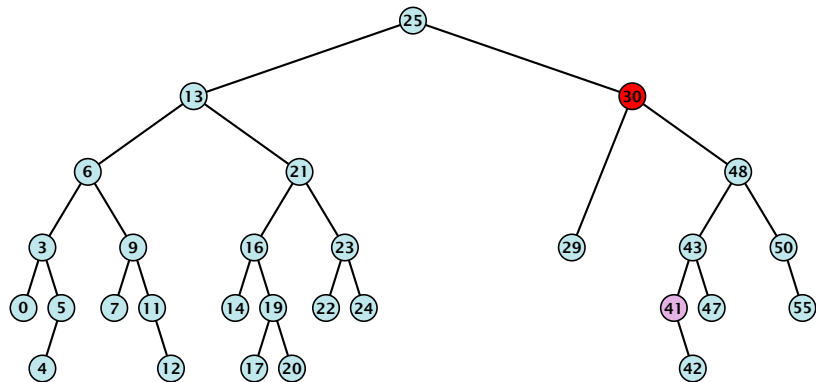


Case 3:

Element has two children

- ▶ Find the successor of the element
- ▶ Splice successor out of the tree
- ▶ Replace content of element by content of successor

Binary Search Trees: Delete

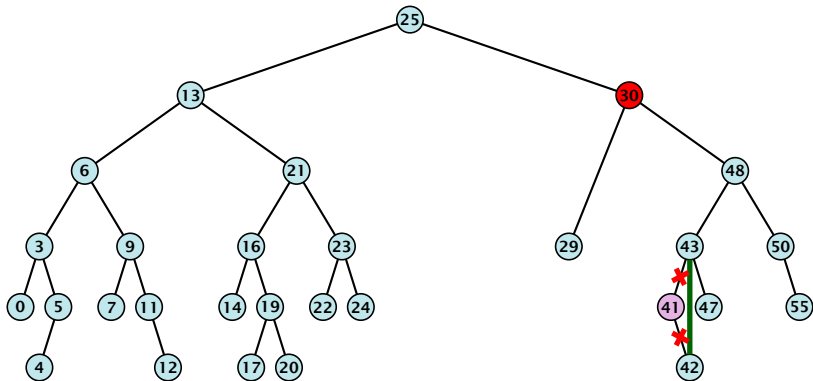


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Binary Search Trees: Delete

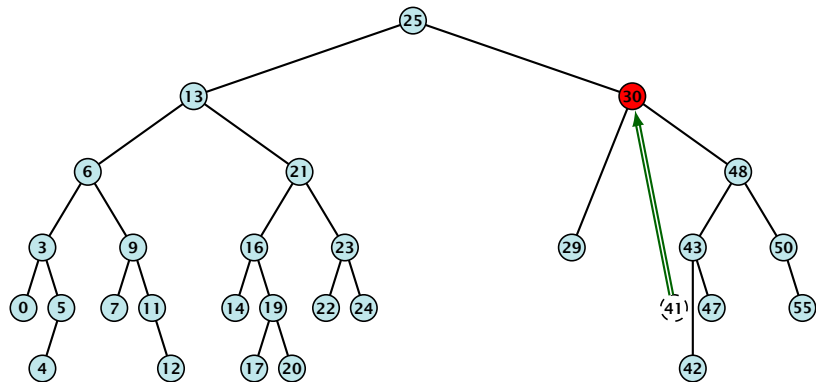


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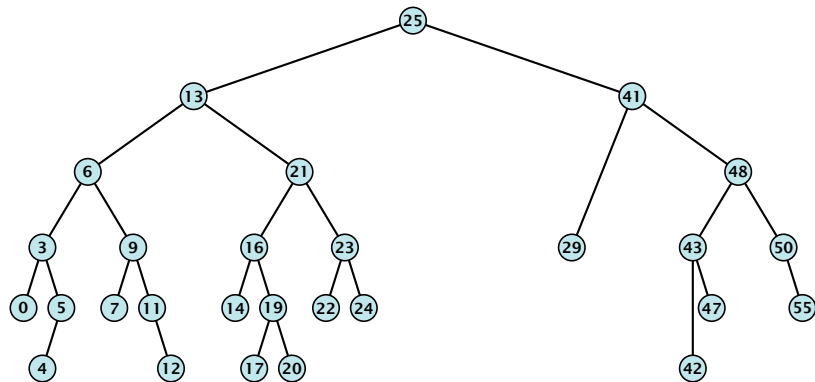


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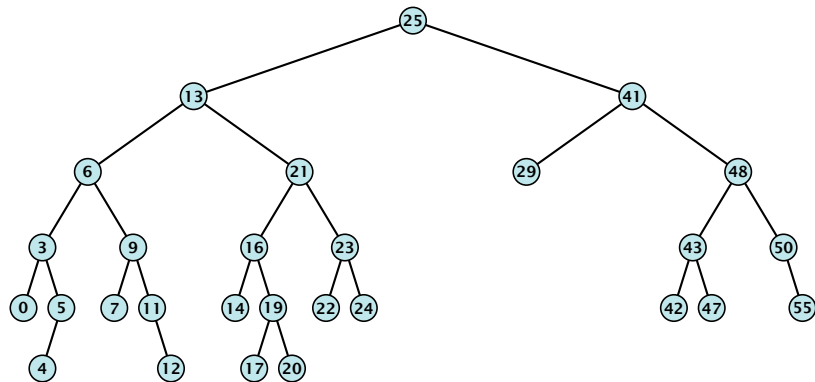


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Binary Search Trees: Delete

Algorithm 9 TreeDelete(z)

```
1: if left[ $z$ ] = null or right[ $z$ ] = null
2:   then  $y \leftarrow z$  else  $y \leftarrow \text{TreeSucc}(z)$ ;   select  $y$  to splice out
3:   if left[ $y$ ]  $\neq$  null
4:     then  $x \leftarrow \text{left}[y]$  else  $x \leftarrow \text{right}[y]$ ;  $x$  is child of  $y$  (or null)
5:   if  $x \neq \text{null}$  then parent[ $x$ ]  $\leftarrow$  parent[ $y$ ];   parent[ $x$ ] is correct
6:   if parent[ $y$ ] = null then
7:     root[ $T$ ]  $\leftarrow x$ 
8:   else
9:     if  $y = \text{left}[\text{parent}[x]]$  then
10:      left[parent[ $y$ ]]  $\leftarrow x$ 
11:    else
12:      right[parent[ $y$ ]]  $\leftarrow x$ 
13:   if  $y \neq z$  then copy  $y$ -data to  $z$ 
```

} fix pointer to x

Balanced Binary Search Trees

All operations on a binary search tree can be performed in time $\mathcal{O}(h)$, where h denotes the height of the tree.

However the height of the tree may become as large as $\Theta(n)$.

Balanced Binary Search Trees

With each insert- and delete-operation perform **local** adjustments to guarantee a height of $\mathcal{O}(\log n)$.

AVL-trees, Red-black trees, Scapegoat trees, 2-3 trees, B-trees, AA trees, Treaps

similar: SPLAY trees.

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