9 van Emde Boas Trees

Dynamic Set Data Structure S:

- $S.\text{insert}(x)$
- $S.\text{delete}(x)$
- $S.\text{search}(x)$
- $S.\text{min}()$
- $S.\text{max}()$
- $S.\text{succ}(x)$
- $S.\text{pred}(x)$

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For this chapter we ignore the problem of storing satellite data:

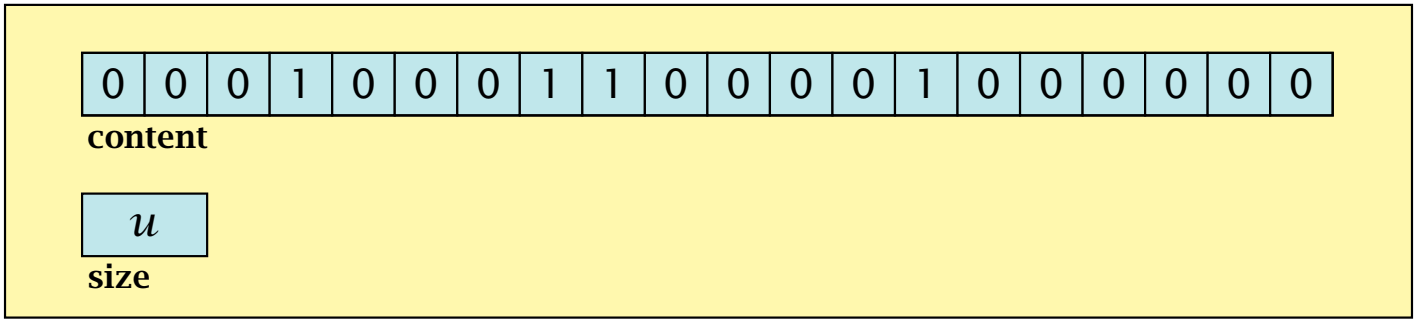
- S**.insert(**x**):** Inserts x into S.
- S**.delete(**x**):** Deletes x from S. Usually assumes that $x \in S$.
- S**.member(**x**):** Returns 1 if $x \in S$ and 0 otw.
- S**.min():** Returns the value of the minimum element in S.
- S**.max():** Returns the value of the maximum element in S.
- S**.succ(**x**):** Returns successor of x in S. Returns null if x is maximum or larger than any element in S. Note that x needs not to be in S.
- S**.pred(**x**):** Returns the predecessor of x in S. Returns null if x is minimum or smaller than any element in S. Note that x needs not to be in S.

9 van Emde Boas Trees

Can we improve the existing algorithms when the keys are from a restricted set?

In the following we assume that the keys are from $\{0, 1, \ldots, u-1\}$, where u denotes the size of the universe.

Implementation 1: Array



one array of u bits

Use an array that encodes the indicator function of the dynamic set.

Implementation 1: Array

Algorithm 19 array.insert(x)

1: content[x] $\leftarrow$ 1;

Algorithm 20 array.delete(x)

1: content[x] $\leftarrow$ 0;

Algorithm 21 array.member(x)

1: **return** content[x];

- Note that we assume that x is valid, i.e., it falls within the array boundaries.
- Obviously(?) the running time is constant.

Implementation 1: Array

Algorithm 22 array.max()

```
1: for (i = size −1; i ≥ 0; i−−) do
2:         if content[i] = 1 then return i;
3: return null;
```

Algorithm 23 array.min()

```
1: for (i = 0; i < size; i++) do
2:         if content[i] = 1 then return i;
3: return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 1: Array

Algorithm 24 array.succ(x)

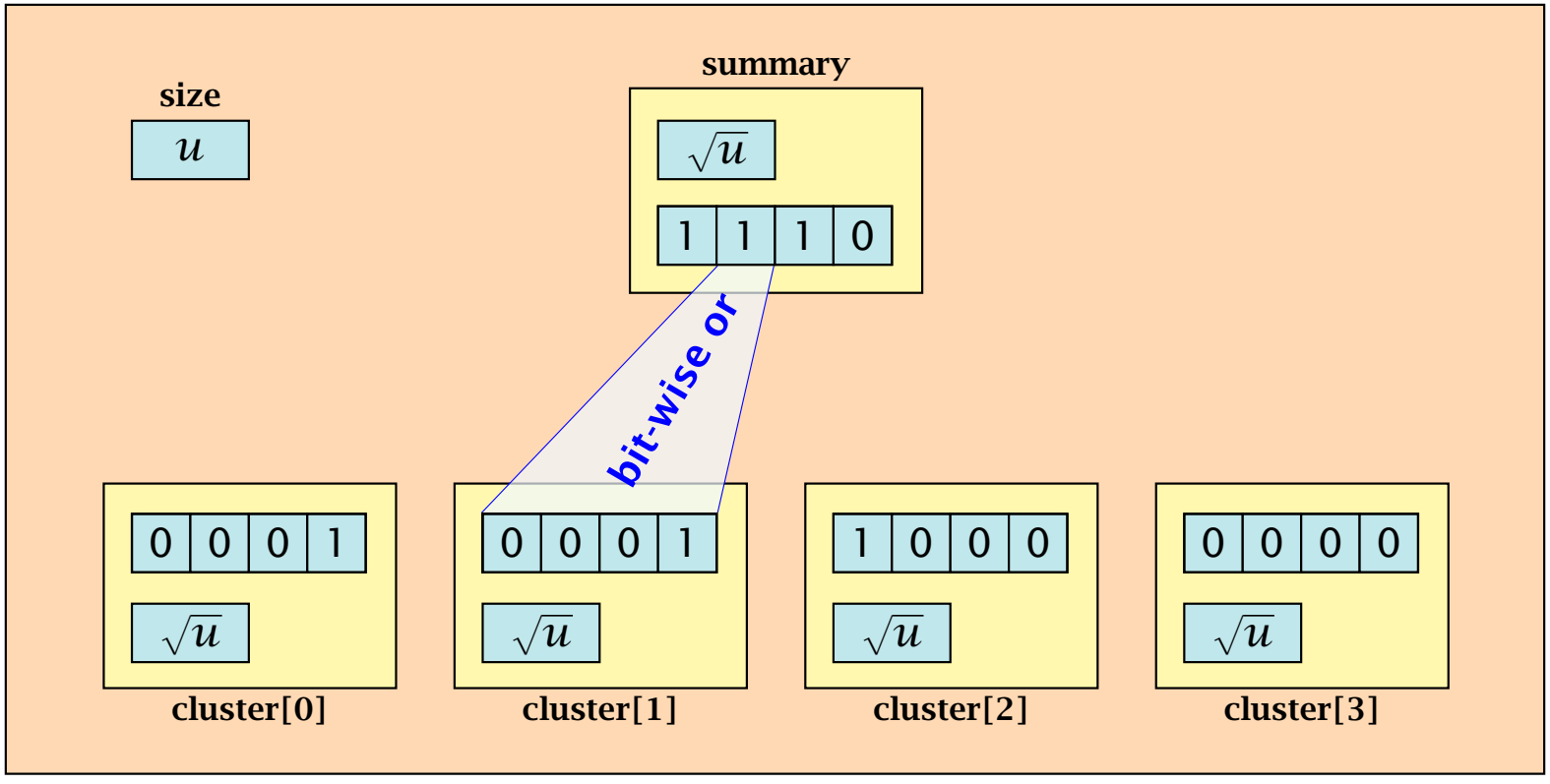
```
1: for (i = x + 1; i < size; i++) do
2:         if content[i] = 1 then return i;
3: return null;
```

Algorithm 25 array.pred(x)

```
1: for (i = x − 1; i ≥ 0; i−−) do
2:         if content[i] = 1 then return i;
3: return null;
```

- Running time is $\mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array



- $\sqrt{u}$ cluster-arrays of $\sqrt{u}$ bits.
- One summary-array of $\sqrt{u}$ bits. The i-th bit in the summary array stores the bit-wise or of the bits in the i-th cluster.

Implementation 2: Summary Array

The bit for a key x is contained in cluster number $\left\lfloor \frac{x}{\sqrt{u}} \right\rfloor$.

Within the cluster-array the bit is at position $x \bmod \sqrt{u}$.

For simplicity we assume that $u = 2^{2k}$ for some $k \geq 1$. Then we can compute the cluster-number for an entry x as $\mathrm{high}(x)$ (the upper half of the dual representation of x) and the position of x within its cluster as $\mathrm{low}(x)$ (the lower half of the dual representation).

Implementation 2: Summary Array

Algorithm 26 member(x)

```
1: return cluster[high(x)].member(low(x));
```

Algorithm 27 insert(x)

```
1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));
```

- The running times are constant, because the corresponding array-functions have constant running times.

Implementation 2: Summary Array

Algorithm 28 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:       summary.delete(high(x));
```

- The running time is dominated by the cost of a minimum computation, which will turn out to be $\mathcal{O}(\sqrt{u})$.

Implementation 2: Summary Array

Algorithm 29 max()

```
1: maxcluster ← summary.max();
2: if maxcluster = null return null;
3: offs ← cluster[maxcluster].max()
4: return maxcluster ∘ offs;
```

Algorithm 30 min()

```
1: mincluster ← summary.min();
2: if mincluster = null return null;
3: offs ← cluster[mincluster].min();
4: return mincluster ∘ offs;
```

The operator $\circ$ stands for the concatenation of two bitstrings. This means if $x = 0111_2$ and $y = 0001_2$ then $x \circ y = 01110001_2$.

- Running time is roughly $2\sqrt{u} = \mathcal{O}(u)$ in the worst case.

Implementation 2: Summary Array

Algorithm 31 succ(x)

```
1: m ← cluster[high(x)].succ(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: succcluster ← summary.succ(high(x));
4: if succcluster ≠ null then
5:         offs ← cluster[succcluster].min();
6:         return succcluster ∘ offs;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 2: Summary Array

Algorithm 32 pred(x)

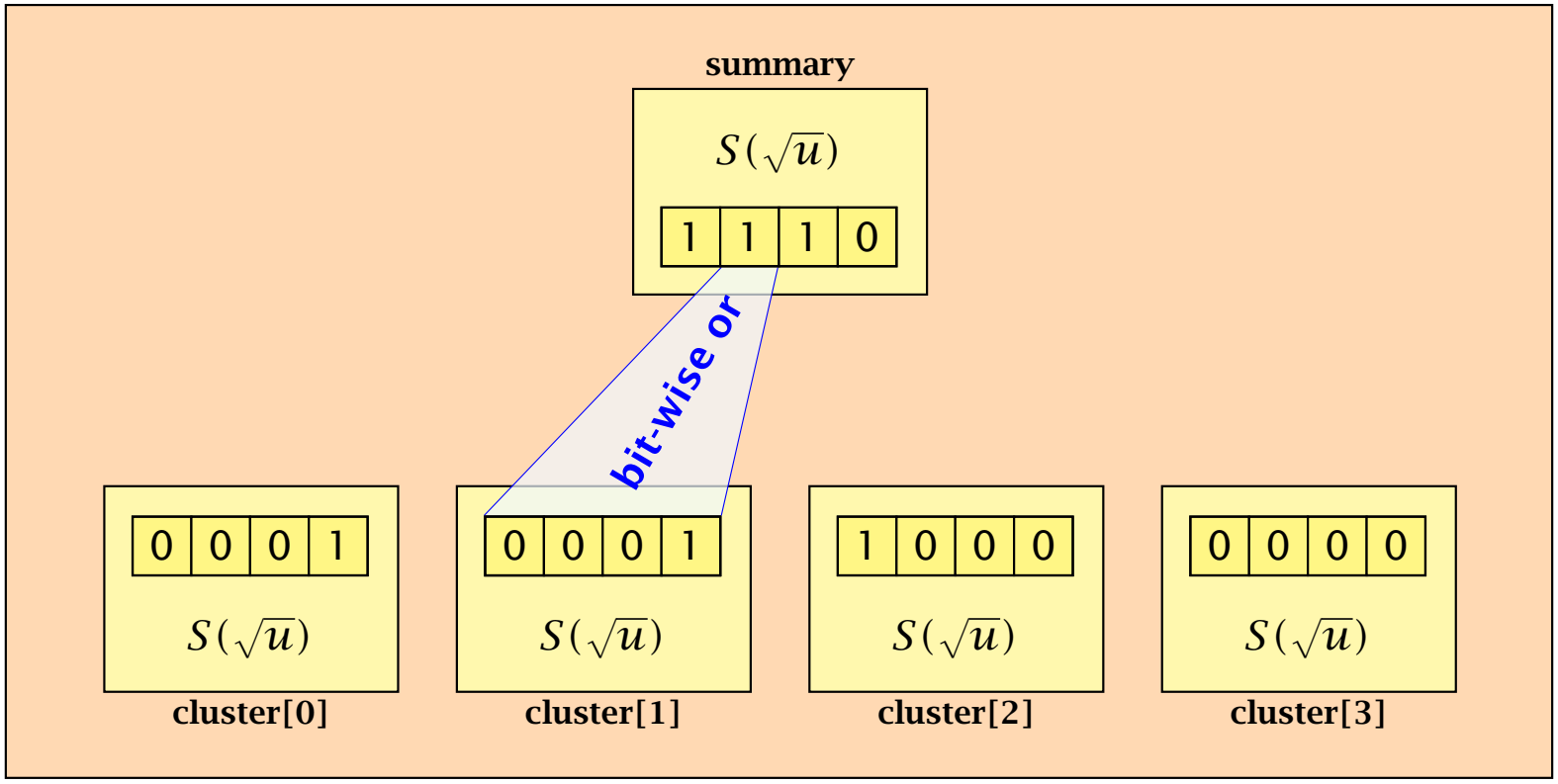
```
1: m ← cluster[high(x)].pred(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: predcluster ← summary.pred(high(x));
4: if predcluster ≠ null then
5:         offs ← cluster[predcluster].max();
6:         return predcluster ∘ offs;
7: return null;
```

- Running time is roughly $3\sqrt{u} = \mathcal{O}(\sqrt{u})$ in the worst case.

Implementation 3: Recursion

Instead of using sub-arrays, we build a recursive data-structure.

$S(u)$ is a dynamic set data-structure representing u bits:



Implementation 3: Recursion

We assume that $u = 2^{2^k}$ for some k.

The data-structure $S(2)$ is defined as an array of 2-bits (end of the recursion).

Implementation 3: Recursion

The code from Implementation 2 can be used unchanged. We only need to redo the analysis of the running time.

Note that in the code we do not need to specifically address the non-recursive case. This is achieved by the fact that an $S(4)$ will contain $S(2)$'s as sub-datastructures, which are arrays. Hence, a call like $\mathrm{cluster}[1].\min()$ from within the data-structure $S(4)$ is not a recursive call as it will call the function $\mathrm{array}.\min()$.

This means that the non-recursive case is been dealt with while initializing the data-structure.

Implementation 3: Recursion

Algorithm 33 member(x)

```
1: return cluster[high(x)].member(low(x));
```

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 34 insert(x)

```
1: cluster[high(x)].insert(low(x));
2: summary.insert(high(x));
```

- $T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 35 delete(x)

```
1: cluster[high(x)].delete(low(x));
2: if cluster[high(x)].min() = null then
3:       summary.delete(high(x));
```

- $T_{\text{del}}(u) = 2T_{\text{del}}(\sqrt{u}) + T_{\min}(\sqrt{u}) + 1.$

Implementation 3: Recursion

Algorithm 36 min()

```
1: mincluster ← summary.min();
2: if mincluster = null return null;
3: offs ← cluster[mincluster].min();
4: return mincluster ∘ offs;
```

- $T_{\min}(u) = 2T_{\min}(\sqrt{u}) + 1$.

Implementation 3: Recursion

Algorithm 37 succ(x)

```
1: m ← cluster[high(x)].succ(low(x))
2: if m ≠ null then return high(x) ∘ m;
3: succcluster ← summary.succ(high(x));
4: if succcluster ≠ null then
5:       offs ← cluster[succcluster].min();
6:       return succcluster ∘ offs;
7: return null;
```

- $T_{\text{succ}}(u) = 2T_{\text{succ}}(\sqrt{u}) + T_{\text{min}}(\sqrt{u}) + 1.$

Implementation 3: Recursion

$T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$:

Set $\ell := \log u$ and $X(\ell) := T_{\text{mem}}(2^\ell)$.Then

$$X(\ell) = T_{\text{mem}}(2^\ell) = T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1$$
$$= T_{\text{mem}}(2^{\frac{\ell}{2}}) + 1 = X(\tfrac{\ell}{2}) + 1 \ .$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\log \ell)$, and hence $T_{\text{mem}}(u) = \mathcal{O}(\log \log u)$.

Implementation 3: Recursion

$$T_{\text{ins}}(u) = 2T_{\text{ins}}(\sqrt{u}) + 1.$$

Set $\ell := \log u$ and $X(\ell) := T_{\text{ins}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\text{ins}}(2^\ell) = T_{\text{ins}}(u) &= 2T_{\text{ins}}(\sqrt{u}) + 1 \\ &= 2T_{\text{ins}}(2^{\frac{\ell}{2}}) + 1 = 2X(\tfrac{\ell}{2}) + 1 \ . \end{aligned}$$

Using Master theorem gives $X(\ell) = \mathcal{O}(\ell)$, and hence $T_{\text{ins}}(u) = \mathcal{O}(\log u)$.

The same holds for $T_{\max}(u)$ and $T_{\min}(u)$.

Implementation 3: Recursion

$$T_{\mathrm{del}}(u) = 2T_{\mathrm{del}}(\sqrt{u}) + T_{\mathrm{min}}(\sqrt{u}) + 1 = 2T_{\mathrm{del}}(\sqrt{u}) + \Theta(\log(u)).$$

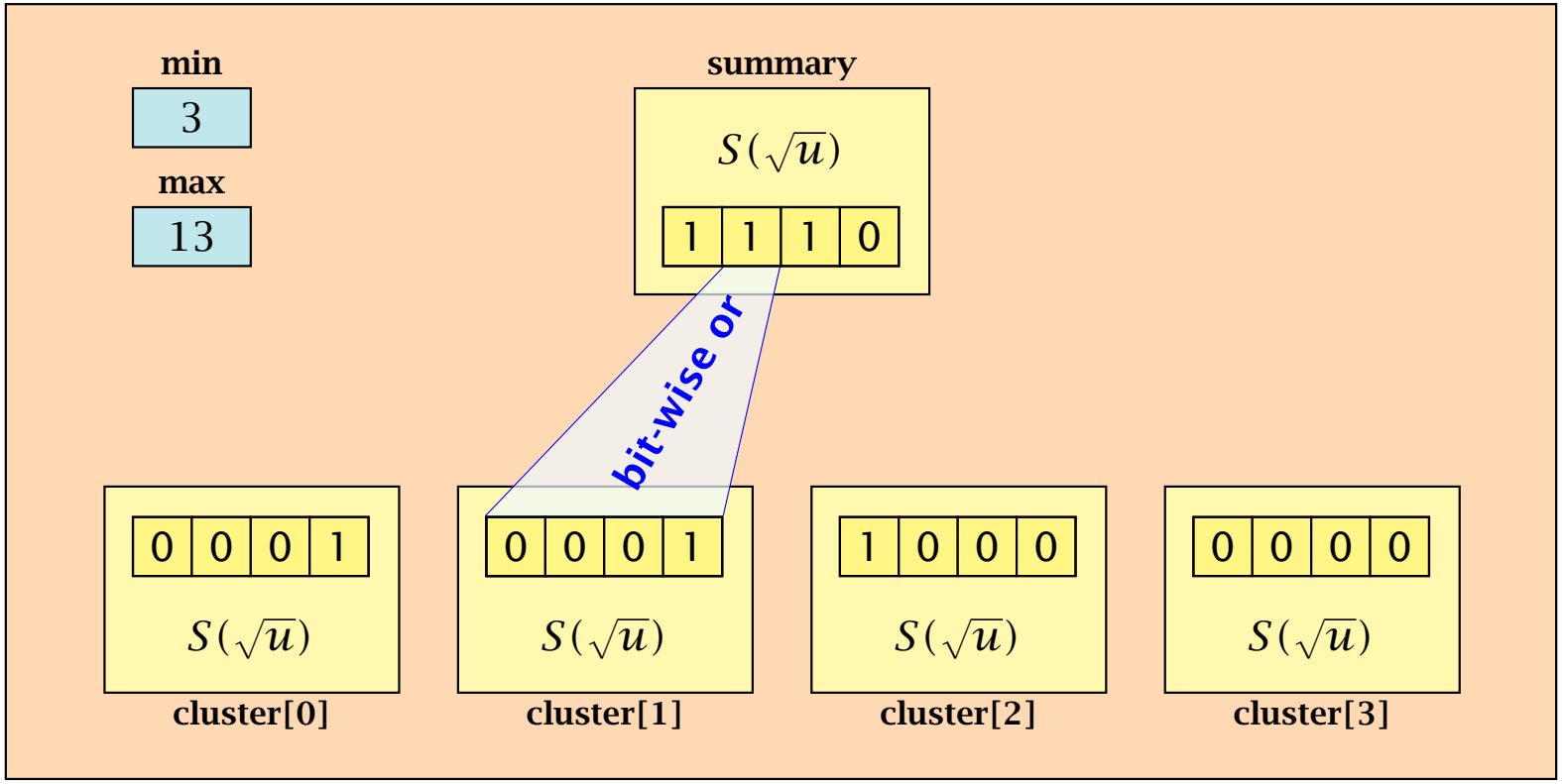
Set $\ell := \log u$ and $X(\ell) := T_{\mathrm{del}}(2^\ell)$. Then

$$\begin{aligned} X(\ell) = T_{\mathrm{del}}(2^\ell) = T_{\mathrm{del}}(u) &= 2T_{\mathrm{del}}(\sqrt{u}) + \Theta(\log u) \\ &= 2T_{\mathrm{del}}(2^{\frac{\ell}{2}}) + \Theta(\ell) = 2X(\tfrac{\ell}{2}) + \Theta(\ell) \ . \end{aligned}$$

Using Master theorem gives $X(\ell) = \Theta(\ell \log \ell)$, and hence $T_{\mathrm{del}}(u) = \mathcal{O}(\log u \log\log u)$.

The same holds for $T_{\mathrm{pred}}(u)$ and $T_{\mathrm{succ}}(u)$.

Implementation 4: van Emde Boas Trees



- The bit referenced by $\min$ is not set within sub-datastructures.
- The bit referenced by $\max$ is set within sub-datastructures (if $\max \neq \min$).

Implementation 4: van Emde Boas Trees

Advantages of having max/min pointers:

- Recursive calls for $\min$ and $\max$ are constant time.
- $\min = \text{null}$ means that the data-structure is empty.
- $\min = \max \neq \text{null}$ means that the data-structure contains exactly one element.
- We can insert into an empty datastructure in constant time by only setting $\min = \max = x$.
- We can delete from a data-structure that just contains one element in constant time by setting $\min = \max = \text{null}$.

Implementation 4: van Emde Boas Trees

Algorithm 38 max()

```
1: return max;
```

Algorithm 39 min()

```
1: return min;
```

- Constant time.

Implementation 4: van Emde Boas Trees

Algorithm 40 member(x)

1: **if** $x = \min$ **then return** 1; // TRUE
2: **return** cluster[high(x)].member(low(x));

- $T_{\text{mem}}(u) = T_{\text{mem}}(\sqrt{u}) + 1 \Longrightarrow T(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 41 succ(x)

```
 1: if min ≠ null ∧ x < min then return min;
 2: maxincluster ← cluster[high(x)].max();
 3: if maxincluster ≠ null ∧ low(x) < maxincluster then
 4:         offs ← cluster[high(x)].succ(low(x));
 5:         return high(x) ∘ offs;
 6: else
 7:         succcluster ← summary.succ(high(x));
 8:         if succcluster = null then return null;
 9:         offs ← cluster[succcluster].min();
10:         return succcluster ∘ offs;
```

- $T_{\text{succ}}(u) = T_{\text{succ}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{succ}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Algorithm 42 insert(x)

```
 1: if min = null then
 2:       min = x; max = x;
 3: else
 4:       if x < min then exchange x and min;
 5:       if cluster[high(x)].min = null; then
 6:             summary.insert(high(x));
 7:             cluster[high(x)].insert(low(x));
 8:       else
 9:             cluster[high(x)].insert(low(x));
10:       if x > max then max = x;
```

- $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1 \Longrightarrow T_{\text{ins}}(u) = \mathcal{O}(\log\log u)$.

Implementation 4: van Emde Boas Trees

Note that the recusive call in Line 7 takes constant time as the if-condition in Line 5 ensures that we are inserting in an empty sub-tree.

The only non-constant recursive calls are the call in Line 6 and in Line 9. These are mutually exclusive, i.e., only one of these calls will actually occur.

From this we get that $T_{\text{ins}}(u) = T_{\text{ins}}(\sqrt{u}) + 1$.

Implementation 4: van Emde Boas Trees

- **Assumes that x is contained in the structure.**

Algorithm 43 delete(x)

```
1: if min = max then
2:     min = null; max = null;
3: else
4:     if x = min then                          find new minimum
5:         firstcluster ← summary.min();
6:         offs ← cluster[firstcluster].min();
7:         x ← firstcluster ∘ offs;
8:         min ← x;
9:     cluster[high(x)].delete(low(x));         delete
```

continued...

Implementation 4: van Emde Boas Trees

Algorithm 43 delete(x)

...continued — fix maximum

```
10:     if cluster[high(x)].min() = null then
11:         summary.delete(high(x));
12:         if x = max then
13:             summax ← summary.max();
14:             if summax = null then max ← min;
15:             else
16:                 offs ← cluster[summax].max();
17:                 max ← summax ∘ offs
18:     else
19:         if x = max then
20:             offs ← cluster[high(x)].max();
21:             max ← high(x) ∘ offs;
```

Implementation 4: van Emde Boas Trees

Note that only one of the possible recusive calls in Line 9 and Line 11 in the deletion-algorithm may take non-constant time.

To see this observe that the call in Line 11 only occurs if the cluster where x was deleted is now empty. But this means that the call in Line 9 deleted the last element in $\mathrm{cluster}[\mathrm{high}(x)]$. Such a call only takes constant time.

Hence, we get a recurrence of the form

$$T_{\mathrm{del}}(u) = T_{\mathrm{del}}(\sqrt{u}) + c \ .$$

This gives $T_{\mathrm{del}}(u) = \mathcal{O}(\log\log u)$.

9 van Emde Boas Trees

Space requirements:

- The space requirement fulfills the recurrence

$$S(u) = (\sqrt{u}+1)S(\sqrt{u}) + \mathcal{O}(\sqrt{u}) \ .$$

- Note that we cannot solve this recurrence by the Master theorem as the branching factor is not constant.
- One can show by induction that the space requirement is $S(u) = \mathcal{O}(u)$. Exercise.