Analysis

- We will show that after at most n reweighting steps the size of the maximum matching can be increased by finding an augmenting path.
- This gives a polynomial running time.

Analysis

How do we find S?

- Start on the left and compute an alternating tree, starting at any free node u.
- If this construction stops, there is no perfect matching in the tight subgraph (because for a perfect matching we need to find an augmenting path starting at u).
- The set of even vertices is on the left and the set of odd vertices is on the right and contains all neighbours of even nodes.
- All odd vertices are matched to even vertices. Furthermore, the even vertices additionally contain the free vertex u. Hence, $|V_{\text{odd}}| = |\Gamma(V_{\text{even}})| < |V_{\text{even}}|$, and all odd vertices are saturated in the current matching.

Analysis

- The current matching does not have any edges from V_{odd} to outside of $L \setminus V_{\text{even}}$ (edges that may possibly deleted by changing weights).
- After changing weights, there is at least one more edge connecting V_{even} to a node outside of V_{odd}. After at most n reweights we can do an augmentation.
- A reweighting can be trivially performed in time $\mathcal{O}(n^2)$ (keeping track of the tight edges).
- An augmentation takes at most $\mathcal{O}(n)$ time.
- In total we otain a running time of $\mathcal{O}(n^4)$.
- A more careful implementation of the algorithm obtains a running time of $\mathcal{O}(n^3)$.

A Fast Matching Algorithm

Algorithm 54 Bimatch-Hopcroft-Karp(G)

1: $M \leftarrow \emptyset$
2: **repeat**
3: let $\mathcal{P} = \{P_1, \dots, P_k\}$ be maximal set of
4: vertex-disjoint, shortest augmenting path w.r.t. M.
5: $M \leftarrow M \oplus (P_1 \cup \dots \cup P_k)$
6: **until** $\mathcal{P} = \emptyset$
7: **return** M

We call one iteration of the repeat-loop a phase of the algorithm.

Analysis

Lemma 98

Given a matching M and a maximal matching M^* there exist $|M^*| - |M|$ vertex-disjoint augmenting path w.r.t. M.

Proof:

- Similar to the proof that a matching is optimal iff it does not contain an augmenting paths.
- Consider the graph $G = (V, M \oplus M^*)$, and mark edges in this graph blue if they are in M and red if they are in M^*.
- The connected components of G are cycles and paths.
- The graph contains $k \stackrel{\text{def}}{=} |M^*| - |M|$ more red edges than blue edges.
- Hence, there are at least k components that form a path starting and ending with a blue edge. These are augmenting paths w.r.t. M.

Analysis

- Let $P_1, \dots, P_k$ be a maximal collection of vertex-disjoint, shortest augmenting paths w.r.t. M (let $\ell = |P_i|$).
- $M' \stackrel{\text{def}}{=} M \oplus (P_1 \cup \dots \cup P_k) = M \oplus P_1 \oplus \dots \oplus P_k$.
- Let P be an augmenting path in M'.

Lemma 99

The set $A \stackrel{\text{def}}{=} M \oplus (M' \oplus P) = (P_1 \cup \dots \cup P_k) \oplus P$ contains at least $(k+1)\ell$ edges.

Analysis

Proof.

- The set describes exactly the symmetric difference between matchings M and $M' \oplus P$.
- Hence, the set contains at least $k+1$ vertex-disjoint augmenting paths w.r.t. M as $|M'| = |M| + k + 1$.
- Each of these paths is of length at least ℓ.

Analysis

Lemma 100

P is of length at least $\ell + 1$. This shows that the length of a shortest augmenting path increases between two phases of the Hopcroft-Karp algorithm.

Proof.

- If P does not intersect any of the $P_1, \dots, P_k$, this follows from the maximality of the set $\{P_1, \dots, P_k\}$.
- Otherwise, at least one edge from P coincides with an edge from paths $\{P_1, \dots, P_k\}$.
- This edge is not contained in A.
- Hence, $|A| \le k\ell + |P| - 1$.
- The lower bound on $|A|$ gives $(k+1)\ell \le |A| \le k\ell + |P| - 1$, and hence $|P| \ge \ell + 1$.

Analysis

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M + |\frac{|V|}{\ell+1}$.

Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

Analysis

Lemma 101
The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

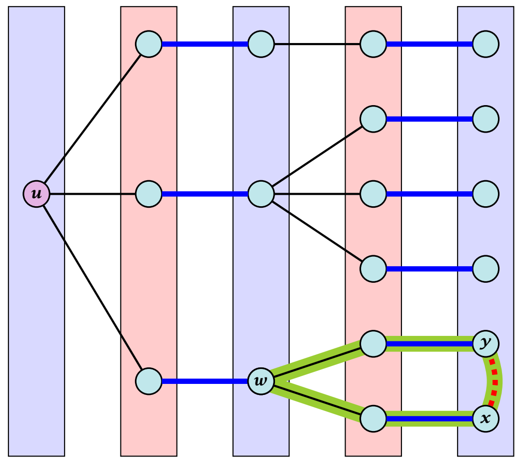
Proof.
- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

Analysis

Lemma 102
One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w path is called the stem of the blossom.