Analysis

If the shortest augmenting path w.r.t. a matching M has ℓ edges then the cardinality of the maximum matching is of size at most $|M + |\frac{|V|}{\ell+1}$.

Proof.
The symmetric difference between M and M^* contains $|M^*| - |M|$ vertex-disjoint augmenting paths. Each of these paths contains at least $\ell + 1$ vertices. Hence, there can be at most $\frac{|V|}{\ell+1}$ of them.

Analysis

Lemma 101
The Hopcroft-Karp algorithm requires at most $2\sqrt{|V|}$ phases.

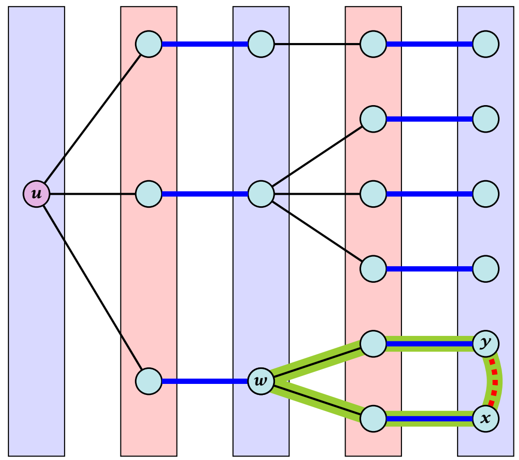
Proof.
- After iteration $\lfloor\sqrt{|V|}\rfloor$ the length of a shortest augmenting path must be at least $\lfloor\sqrt{|V|}\rfloor + 1 \geq \sqrt{|V|}$.
- Hence, there can be at most $|V|/(\sqrt{|V|} + 1) \leq \sqrt{|V|}$ additional augmentations.

Analysis

Lemma 102
One phase of the Hopcroft-Karp algorithm can be implemented in time $\mathcal{O}(m)$.

How to find an augmenting path?

Construct an alternating tree.



even nodes
odd nodes

Case 4:
y is already contained in T as an even vertex

can't ignore y

The cycle $w \leftrightarrow y - x \leftrightarrow w$ is called a blossom.
w is called the base of the blossom (even node!!!).
The path u-w path is called the stem of the blossom.

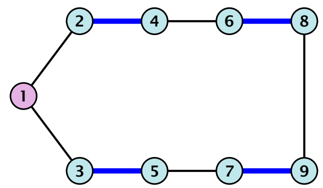
Flowers and Blossoms

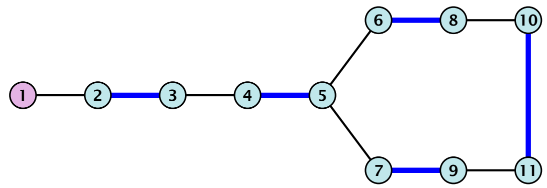
Definition 103

A flower in a graph $G = (V, E)$ w.r.t. a matching M and a (free) root node r, is a subgraph with two components:

- A stem is an even length alternating path that starts at the root node r and terminates at some node w. We permit the possibility that $r = w$ (empty stem).
- A blossom is an odd length alternating cycle that starts and terminates at the terminal node w of a stem and has no other node in common with the stem. w is called the base of the blossom.

Flowers and Blossoms





Flowers and Blossoms

Properties:

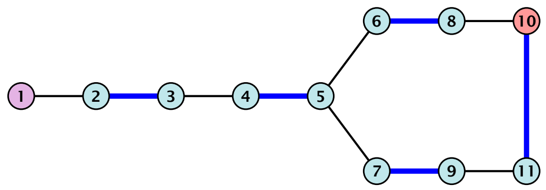
1. A stem spans $2\ell + 1$ nodes and contains ℓ matched edges for some integer $\ell \geq 0$.
2. A blossom spans $2k + 1$ nodes and contains k matched edges for some integer $k \geq 1$. The matched edges match all nodes of the blossom except the base.
3. The base of a blossom is an even node (if the stem is part of an alternating tree starting at r).

Flowers and Blossoms

Properties:

4. Every node x in the blossom (except its base) is reachable from the root (or from the base of the blossom) through two distinct alternating paths; one with even and one with odd length.
5. The even alternating path to x terminates with a matched edge and the odd path with an unmatched edge.

Flowers and Blossoms

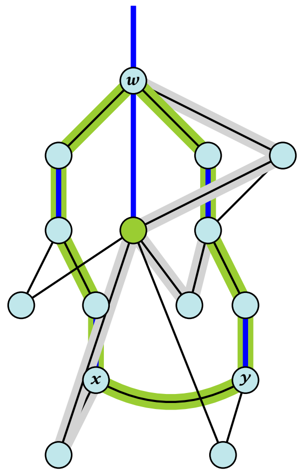


When during the alternating tree construction we discover a blossom B we replace the graph G by $G' = G/B$, which is obtained from G by contracting the blossom B.

- Delete all vertices in B (and its incident edges) from G.
- Add a new (pseudo-)vertex b. The new vertex b is connected to all vertices in $V \setminus B$ that had at least one edge to a vertex from B.

Shrinking Blossoms

- Edges of T that connect a node u not in B to a node in B become tree edges in T' connecting u to b.
- Matching edges (there is at most one) that connect a node u not in B to a node in B become matching edges in M'.
- Nodes that are connected in G to at least one node in B become connected to b in G'.



Algorithm 55 search(r, found)

set $\bar{A}(i) \leftarrow A(i)$ for all nodes i
$\mathit{found} \leftarrow$ false
unlabel all nodes;
give an even label to r and initialize $\mathit{list} \leftarrow \{r\}$
while $\mathit{list} \neq \emptyset$ **do**
    delete a node i from list
    examine(i, found)
    if $\mathit{found} =$ true **then**
        return

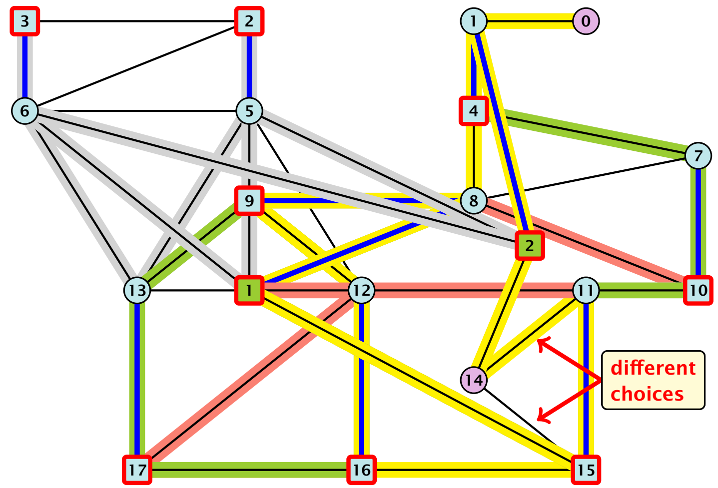
Algorithm 56 examine$(i, found)$

```
for all j ∈ Ā(i) do
    if j is even then contract(i, j) and return
    if j is unmatched then
        q ← j;
        pred(q) ← i;
        found ← true;
        return
    if j is matched and unlabeled then
        pred(j) ← i;
        pred(mate(j)) ← j;
```


Algorithm 57 contract(i, j)

```
trace pred-indices of i and j to identify a blossom B
create new node b and set Ā(b) ← ∪_{x∈B} Ā(k)
label b even and add to list
update Ā(j) ← Ā(j) ∪ {b} for each j ∈ Ā(b)
form a circular doubly linked list of nodes in B
delete nodes in B from the graph
```


Example: Blossom Algorithm



Assume that we have contracted a blossom B w.r.t. a matching M whose base is w. We created graph $G' = G/B$ with pseudonode b. Let M' be the matching in the contracted graph.

Lemma 104

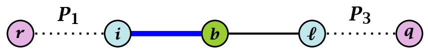
If G' contains an augmenting path p' starting at r (or the pseudo-node containing r) w.r.t. to the matching M' then G contains an augmenting path starting at r w.r.t. matching M.

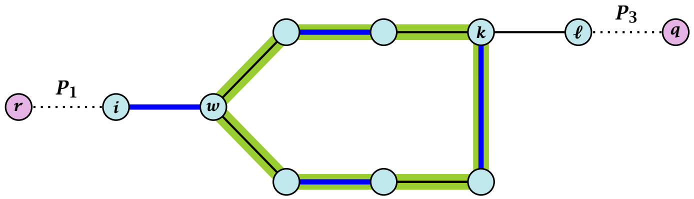
Proof.

If p' does not contain b it is also an augmenting path in G.

Case 1: non-empty stem

- Next suppose that the stem is non-empty.



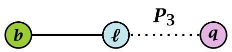


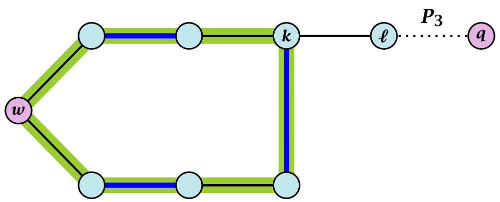
- After the expansion ℓ must be incident to some node in the blossom. Let this node be k.
- If $k \neq w$ there is an alternating path P_2 from w to k that ends in a matching edge.
- $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.
- If $k = w$ then $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$ is an alternating path.

Proof.

Case 2: empty stem

- If the stem is empty then after expanding the blossom, $w = r$.





- The path $r \circ P_2 \circ (k, \ell) \circ P_3$ is an alternating path.

Lemma 105

If G contains an augmenting path P from r to q w.r.t. matching M then G' contains an augmenting path from r (or the pseudo-node containing r) to q w.r.t. M'.

Proof.

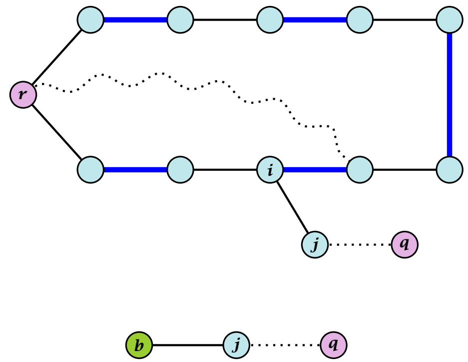
- If P does not contain a node from B there is nothing to prove.
- We can assume that r and q are the only free nodes in G.

Case 1: empty stem

Let i be the last node on the path P that is part of the blossom.

P is of the form $P_1 \circ (i,j) \circ P_2$, for some node j and (i,j) is unmatched.

$(b,j) \circ P_2$ is an augmenting path in the contracted network.



Case 2: non-empty stem

Let P_3 be alternating path from r to w. Define $M_+ = M \oplus P_3$.

In M_+, r is matched and w is unmatched.

G must contain an augmenting path w.r.t. matching M_+, since M and M_+ have same cardinality.

This path must go between w and q as these are the only unmatched vertices w.r.t. M_+.

For M'_+ the blossom has an empty stem. Case 1 applies.

G' has an augmenting path w.r.t. M'_+. It must also have an augmenting path w.r.t. M', as both matchings have the same cardinality.

This path must go between r and q.

Example: Blossom Algorithm