Guaranteed Stable Projection-Based Model Reduction for Indefinite and Unstable Linear Systems *

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* B.N.Bond,L.Daniel, IEEE Trans,ICCAD,Nov.2008,P.728-735

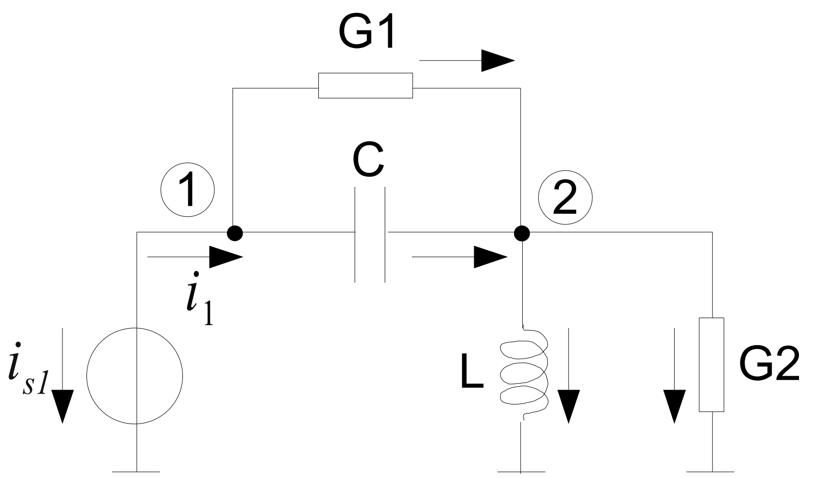
overview

- 1 system model
- 2 properties
- 3 great Russian researchers
- Presented tool
  - Projection matrix
  - Stability constraints
  - Solving the LMI(Linear Matrix Inequality)
- Experimental results
- Conclusion

1 System Model

System model

- Physical system(VLSI application) → field-solvers / parasitic extractors
- warm-up: RLC example



$$C \cdot \frac{\partial u(t)}{\partial t} = \frac{\partial Q(t)}{\partial t} = i(t)$$

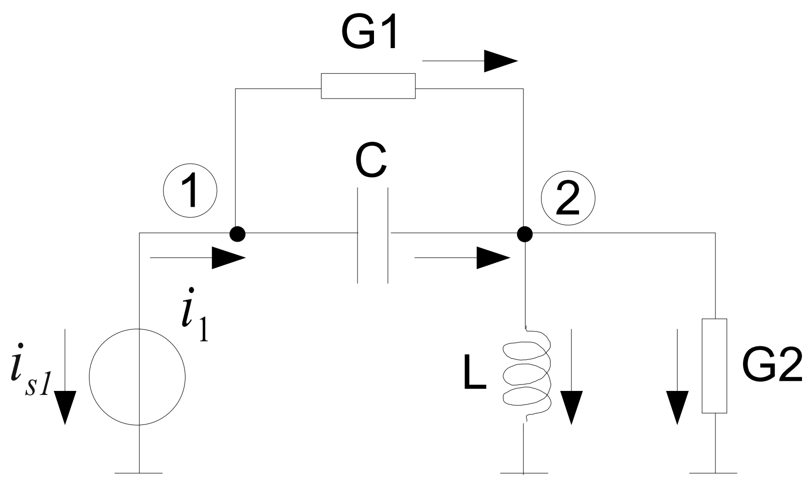
$$\boxed{C\dot{u} = i}$$

$$L\dot{i} = u$$

- Tackle the model: Modified Nodal Analysis
  - Find nodal
  - Kirchhoff's Laws

1 system Model

- Tackle the model: Kirchhoff's Laws



$$C(\dot{v}_1-\dot{v}_2) + G_1(v_1-v_2) + i_{s1} = 0$$

$$C(\dot{v}_2-\dot{v}_1) + G_1(v_2-v_1) + G_2 v_2 + i_L = 0$$

$$L\dot{i}_L - v_2 = 0$$

$$-v_1 = -u_{s1}$$

$$i_1 = -i_{s1}$$

$$\begin{bmatrix} C & -C & 0 & 0 \\ -C & C & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i}_L \\ \dot{i}_{s1} \end{bmatrix} = - \begin{bmatrix} G_1 & -G_1 & 0 & 1 \\ -G_1 & G_1+G_2 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i_L \\ i_{s1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} u_1 \end{bmatrix}$$

1 system Model

- Linear descriptor system

$$E\dot{x} = Ax + Bu, \quad y = C^T x$$

- Transfer function *Tr(s):* through Laplace transformation

$$sEX = AX + BU \longrightarrow (sE - A)X = BU \longrightarrow X = (sE - A)^{-1} BU$$

$$Y = C^T X = C^T (sE - A)^{-1} BU$$
$$Tr(s) = C^T (sE - A)^{-1} B$$

1 System Model - When theory meets practice

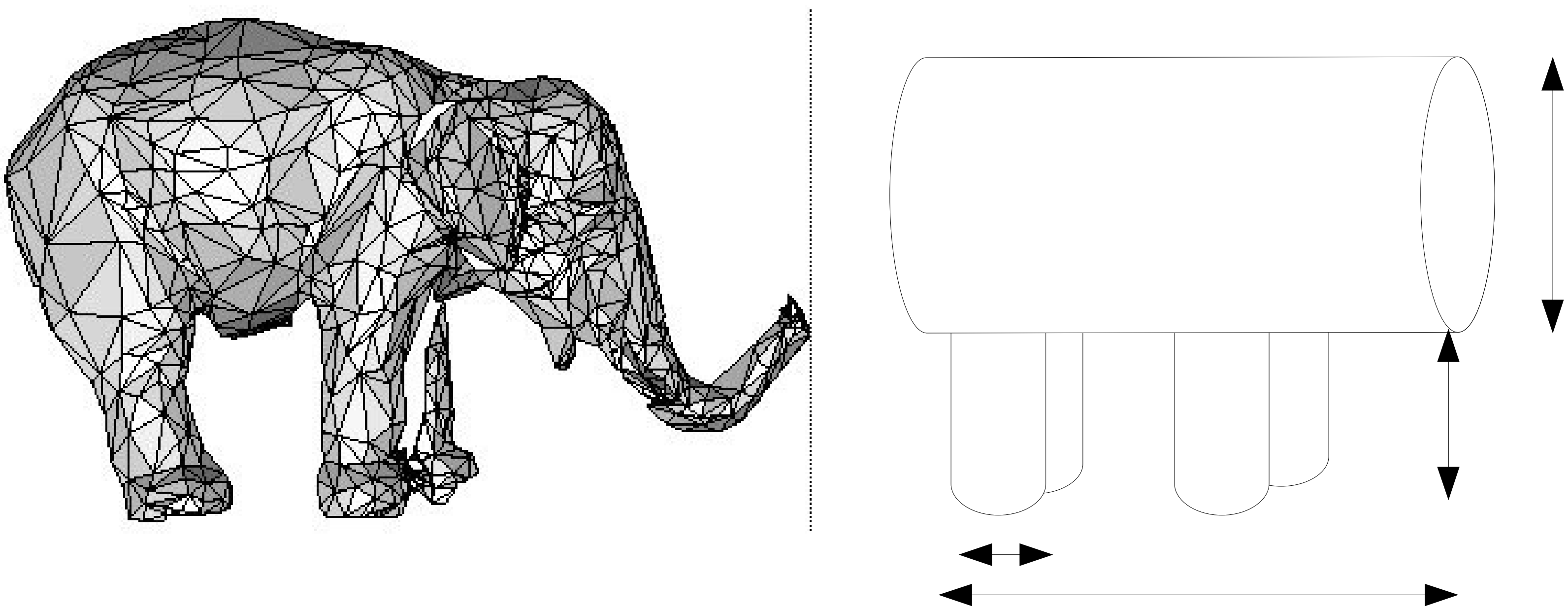
- Feasibility v.s. circuit scale
  - “It is the mark of an educated mind to rest **satisfied** with the degree of precision which the nature of the subject admits and not to seek exactness **where only an approxima-tion is possible.** ”

    \- **Aristotle** (around 350 B.C.)

➔ Approximation by model order reduction

1 System Model - Model Reduction

- Example
  - Task: to weight an elephant
  - Directly weight: put the real elephant on the balance
  - Approximation: extract the most deciding parameters



1 System Model - Model Reduction

- Eigenvalue approximation for M

$$M = U \Lambda V^T$$

  - Choose r largest eigenvalues for the new approximating matrix with rank r

$$\tilde{M} = U \tilde{\Lambda} V^T$$

- Minimal distance to the original matrix by Frobenius norm

$$\|M - \tilde{M}\|_p = \|\Delta M\|_p = (\sum_{j=1}^{c} \sum_{i=1}^{c} |\Delta m_{ij}|)^{1/p}$$

- BUT: eigenvalue decomposition too luxurious

1 System Model - Model Reduction

- Moment matching
  - Concept: Taylor expansion of *Tr(s)*

$$Tr(s)=M_0+M_1 s+M_2 s^2+\dots$$

k-th moment:
k-th coefficients of the Taylor expansion

- Construction of *moment*

$$E\dot{x}=Ax+Bu,\quad y=C^T x \xrightarrow[R=-A^{-1}B]{K=A^{-1}E} M_k=C^T K^k R$$

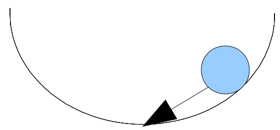
- Contribution of *moment*
  - *s* corresponds to the frequency; the lower order terms dominate the accuracy of Tr(s) => to be approximated

2 properties - stability/passivity

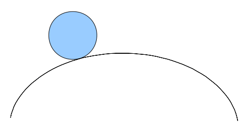
Stability

$$M(s)=L^{-1}(m(t))$$

$$\|m(t)-m_0\|<\epsilon$$



- asymptotic stability



$$\lim m(t)=0\,;\ when\ \ t\to 0$$

Passivity:

- incapable of generating energy of *moment*
- stable but non-passive system could **produce unstable system** when interconnected to stable and passive system
- *Implies stability*

3 great Russian researchers

- ***A.M.Lyapunov***
  - **Lyapunov function**: conditions for a system to be stable and passive
- ***A.N.Krylov***
  - **Krylov Subspace** for picking the basis of the projection matrix to match the desired moments → enforce accuracy
- ***B.G.Galerkin***
  - **Galerkin projection:**
    congruence transformation able to preserve the symmetry and definiteness of the original system

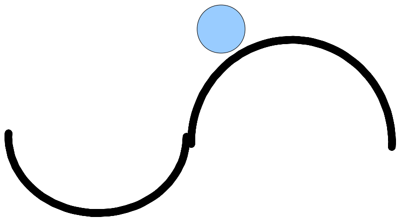
Lyapunov Function

- **Form:**
  - stability: condition in 'mark2'

$$L(x) \geq 0 \ ; \ equal \ only \ when \ x = 0$$

$$\frac{\partial L(x)}{\partial t} \leq 0 \ ; \ equal \ only \ when \ x = 0 \ (for \ asymptotic \ stability)$$

  - passivity: storage func., incorporating inputs
- **Functionality:** its existence verifies the stability[orig.ref[11][12]]



Krylov subspace

- **Form:** Krylov subspace: definition and application

$$Kr(K, R, q) = colspan[R, KR, K^2R, \ldots, K^qR]$$

$$colspan(V) = Kr(K, R, q)$$

$$V^T V = I$$

$$K = A^{-1}E$$

$$R = -A^{-1}B$$

- **Functionality**: for constructing the projection matrix
  - q accurate moments
  - avoid matrix-matrix operation in finding the eigenvalues or solving linear system

Galerkin Projection – congruence transformation

- **Form:** projection pair (V, V)

$$V^T E V \dot{z} = V^T A V z + V^T B u, \quad y = (C^T) V z$$

$$E_{vv} = V^T E V \quad A_{vv} = V^T A V$$

$$B_{v0} = V^T B \quad (C^T)_v = C^T V$$

Congruence transformation

$$E_{vv} \dot{z} = A_{vv} z + B_{v0} u, \quad y = (C^T)_v z$$

- **Functionality:**
  - Order reduced

Galerkin Projection – congruence transformation

- **Functionality:**
  - For system with E and -(A+A^T) SPD, such as in modeling RLC network: automatically preserve the definitiveness[ref.6]

- **Cons**:
  - limited application: extracted large system in VLSI applications are non symmetric and indefinite
  - sacrifice accuracy for stability

Krylov and Galerkin: Moments matching

- Order reduced by (V,V) projection

$$E\dot{x} = Ax + Bu, \quad y = C^T x$$

$$E_{vv}\dot{z} = A_{vv}z + B_{v0}u, \quad y = (C^T)_v z$$

$$Tr(s) = M_0 + M_1 s + M_2 s^2 + \ldots$$

$$\tilde{T}r(s) = \tilde{M}_0 + \tilde{M}_1 s + \tilde{M}_2 s^2 + \ldots$$

$$M_k = C^T K^k R \qquad K = A^{-1}E \qquad R = A^{-1}B$$

$$\tilde{M}_k = (C^T)_v \tilde{K}^k \tilde{R} \qquad \tilde{K} = A_{vv}^{-1}E_{vv} \qquad \tilde{R} = A_{vv}^{-1}B_{v0}$$

It is shown in [PRIMA] that the first q/N moments of the two systems are equal
•mainly by expanding M and simplification

Position check

- A large system → field extractors → a linear descriptor system model
- Stability, passivity → verified by the existence of Lyapunov function
- Order reduction → a projection based way by taking some basis from the Krylov space, then transform the model using Galerkin projection pair.
- But, do we trust our extractors?
  - Original indefinite / unstable model
  - Even physical stable → numerically unstable

Principle of the presented tool

- **Object:** ensure stability and passivity meanwhile being as accurate as possible
  - Sacrifice accuracy for stability
  - Projection pair (V,V) → (U,V) right and left projection pair
    - U for stability and passivity
    - V for accuracy by moment matching

Projection-based Model Reduction

Projection framework

- (V,V) pair -> (U,V) pair
- Non-congruence transformation; also fit for indefinite and unstable models

$$E\dot{x} = Ax + Bu \xrightarrow{x \approx Vz} EV\dot{z} = AVz + Bu$$

$$\downarrow$$

$$E_{uv}\dot{z} = A_{uv}z + B_u u \longleftarrow U^T(EV\dot{z} - AVz - Bu) = 0$$

$$E_{uv} = U^T E V$$

$$A_{uv} = U^T A V$$

$$B_u = U^T B$$

(U,V) pair – *Krylov-subspace*-based construction

Krylov-based, projection matrices constructed as:

$$v_k = ((s_p E - A)^{-1} E)^k (s_p E - A)^{-1} B \quad range(V) \supset v_k$$

$$u_k = C^T (s_q E - A)^{-1} (E (s_q E - A)^{-1})^k \quad range(U) \supset u_k$$

Significance of such constructed U and V:

- Loss of info. → important dynamics
- 0-th to m-th moment matching of the transfer function

at frequency sp and sq

Other methods like POD and TBR:
zeroth moment of the transfer function for multiple frequency points

Consideration on Projection-based Model Reduction?

- How about stability?
  - Galerkin projection keeps the definiteness for the transformed system. Thus, stability and passivity are preserved.
  - Not the case for unstable or indefinite models

↓

Trade accuracy for stability
- Fix V and find U satisfying stability condition

Stability Conditions for U

- A model is stable if its Lyapunov Function exists

$$E\dot{x} = Ax + Bu, \quad y = C^T x$$

$$L(x) = x^T E^T P E x$$

- Property of Lyapunov Function (assume autonomous model with u=0):

$$\frac{\partial L(x)}{\partial t} \leq 0 \qquad \frac{\partial L(x)}{\partial t} = x^T E^T P E \frac{\partial x}{\partial t} + \frac{\partial x^T}{\partial t} E^T P E x = x^T E^T P E\dot{x} + \dot{x}^T E^T P E x$$

$$E\dot{x} \rightarrow Ax \qquad \dot{x}^T E^T \rightarrow (Ax)^T$$

$$\frac{\partial L(x)}{\partial t} = x^T E^T P A x + x^T A^T P E x \leq 0$$

$$E^T P A + A^T P E = -Q_2 \leq 0$$

Stability Conditions for U

- Stability condition for the reduced model

$$E_{uv}\dot{z} = A_{uv} z + B_v u, \quad y = (C^T)_v z \qquad L(z) = z^T E_{uv}^T P E_{uv} z$$

$$E_{uv}^T \hat{P} A_{uv} + A_{uv}^T \hat{P} E_{uv} = -Q_2 \leq 0$$

$$V^T E^T U \hat{P} U^T A V + V^T A^T U \hat{P} U^T E V = -Q_2$$

s.1

$$\hat{U}$$

- Take a look at S.1:
  - Quadratic in U: Try like for E with dimension 1000*1000
  - Let's transform s.1

$$\hat{U}^T A V + V^T A^T \hat{U} = -Q_2$$

$$L(z) = z^T E_{uv}^T P E_{uv} z = z^T V^T E^T U P U^T E V z = z^T \hat{U}^T E V Z$$

s.2

$$L(z) > 0 \quad \Rightarrow \quad \hat{U}^T E V \text{ is } SPD! \quad \Rightarrow \quad \hat{U}^T E V = Q_1$$

Stability Conditions for U

- Equality of s.1 and s.2: a solution to one system could solve the other system after certain transformation
  - from s.1 to s.2

s.1

$$V^T E^T U \hat{P} U^T A V + V^T A^T U \hat{P} U^T E V = -Q_2$$

s.2

$$\hat{U}^T A V + V^T A^T \hat{U} = -Q_2$$

$$\hat{U}^T E V = Q_1$$

$$U, \hat{P}, Q_2 \quad \Longleftrightarrow$$

$$\hat{U} = U \hat{P} U^T E V$$

$$\hat{U}^T E V = (V^T E^T U) \hat{P} (U^T E V) = Q_1$$

Stability Conditions for U

- Equality of s.1 and s.2: a solution to one system could solve the other system after certain transformation
  - from s.2 to s.1

s.2

$$\hat{U}^T A V + V^T A^T \hat{U} = -Q_2$$
$$\hat{U}^T E V = Q_1$$

s.1

$$V^T E^T U \hat{P} U^T A V + V^T A^T U \hat{P} U^T E V = -Q_2$$

$$U, \hat{P}(SPD), Q_2$$

$$\hat{U}, Q_1, Q_2 \Longleftrightarrow$$

$$U = \hat{U}$$
$$\hat{P} = (\hat{U} E V)^{-1} = (Q_1)^{-1}$$

$$V^T E^T \hat{U} (\hat{U} E V)^{-1} \hat{U}^T A V +$$
$$V^T A^T \hat{U} (\hat{U} E V)^{-1} \hat{U}^T E V = -Q_2$$

Stability Conditions for U

- Now, quadratic constraint is replaced by a pair of linear constraints in U → LMI solver

$$\hat{U}^T E V = Q_1$$
$$\hat{U}^T A V + V^T A^T \hat{U} = -Q_2$$

s.2

- By the way
  - for enforce orthogonality

$$U^T E V = E_{uv} = Q_1 \Rightarrow E_{uv}^{-1} E_{uv} = E_{uv}^{-1} Q_1 = I$$

$$E_{uv}^{-1} E_{uv} = \boxed{(U^T E V)^{-1} U^T} E V$$

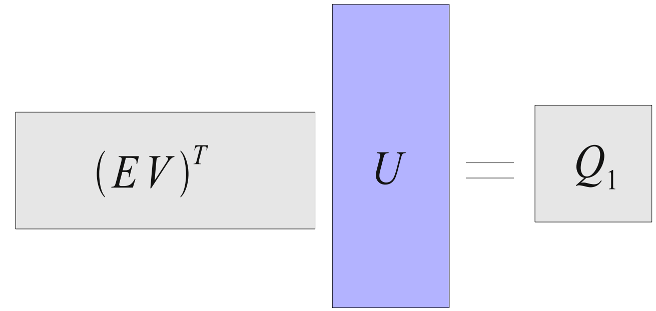
$$U^T$$

  - Also possible fix U and solve for V

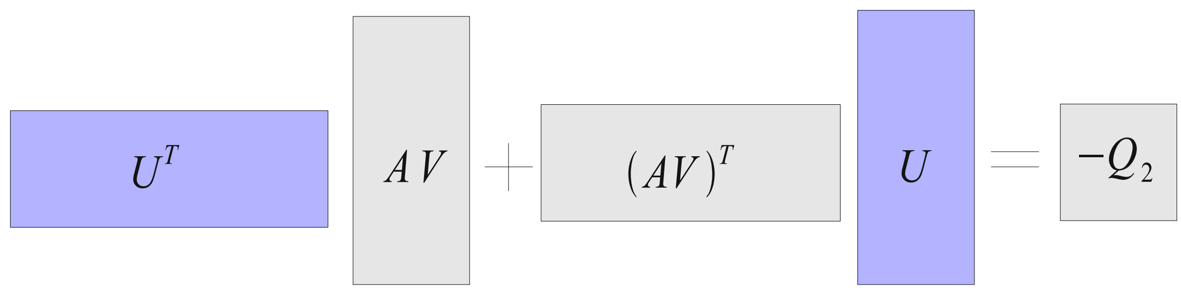
Stabilizing Solutions

- Matrix structure

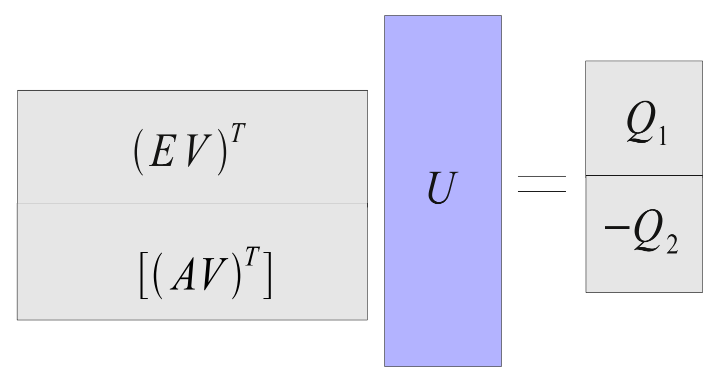
$U^T E V = Q_1$



$U^T A V + V^T A^T U = -Q_2$



- concatenation



Stabilizing Solutions - considerations

- Number of unknowns
  - (Nq) → Computation effort for LMI solvers (Nq)^2
- No dependence on eigenvalues of E,A
  - Enforcing stability
- infinite subspaces exist
  - Find the optimal solution

Solving the LMI – Dependent Constraints

- Trace of engineering: fix N-p rows of variable;
  - fast and cheap
  - adding a perturbation matrix $U_\delta \in \mathbb{R}^{N \times p}$ of the unknowns
- Initial U0 predetermined : presume we already know the answer! And $U = U_0 + U_\delta$ has to satisfy the constraints.

The simplification or trick is $U_\delta \in \mathbb{R}^{N \times p}$ has a significantly low order p, which is artificially chosen. So the LMI constraints is fairly easy to solve.

Solving the LMI – Dependent Constraints

Now: $U = U_0 + U_\delta$ And: $U_\delta = \begin{bmatrix} U_p \\ 0 \end{bmatrix}$

To satisfy: $U^T E_v = Q_1 > 0$

$$\overbrace{\begin{bmatrix} E_{vp}^T & E_{v2}^T \end{bmatrix}}^{E_v^T} \underbrace{\left(\begin{bmatrix} U_0 \end{bmatrix} + \begin{bmatrix} U_p \\ 0 \end{bmatrix} \right)}_{U} = Q_1 \Rightarrow E_{vp}^T U_p = Q_1 - \overbrace{E_v^T U_2}^{\Delta Q_1}$$

Solving the LMI – Dependent Constraints

Similarly, to satisfy: $U^T A_v + A_v^{T} U = -Q_2 < 0$

$$-\Delta Q_2$$

$$U_p^T A_{vp} + A_{vp}^{T} U_p = -Q_2 - A_v^{T} U_0 - A_v^{T} U_0$$

- 2q < p << N
- Q-order LMI, **independent** on N;
  - Unknowns O(p^2) → Cost O(p^4)
- Select only non-zero rows

Optimization over constraints

- Infinite stable projection subspaces spanned by U
  - Optimization problem for accuracy
    - ✔ Be ambitious: start right with U0 for the best accuracy

$$min(\|U-U_0\|)=min(\|U_\delta\|)$$

$$U^T E_v = Q_1 > 0$$
$$U^T A_v + A_v^T U = -Q_2 < 0$$

$\Rightarrow$

$$E_{vp}^T U_\delta = Q_1 - \Delta Q_1$$
$$A_{vp}^T U_\delta + U_\delta^T A_{vp} = -Q_2 - \Delta Q_2$$
$$\Delta Q_1 = V_e^T U_0$$
$$\Delta Q_2 = A_v^T U_0 + U_0^T A_v$$

Final Algorithm 1

- Given E, A, V, U_0
  - Define

$$E_v = EV \qquad A_v = AV$$

$$\Delta Q_1 = E_v^T U_0 \qquad \Delta Q_2 = V_a^T U_0 + U_0^T V_a$$

- p-nonzero Rows selection

$$E_{vp} = sel(E_v, p) \quad A_{vp} = sel(A_v, p)$$

- Optimization formulation

$$\min_{U_\delta, Q_1 > 0, Q_2 > 0} \quad \|U_\delta\|$$
$$s.t. \quad U_\delta^T E_{vp} = Q_1 - \Delta Q_1$$
$$U_\delta^T A_{vp} + U_\delta^T A_{vp}^T = -Q_2 - \Delta Q_2$$

- Adding the perturbation

$$U = U_0 + \Delta U$$

Consideration on this algorithm

- SUB-OPTIMAL w.r.t a certain q

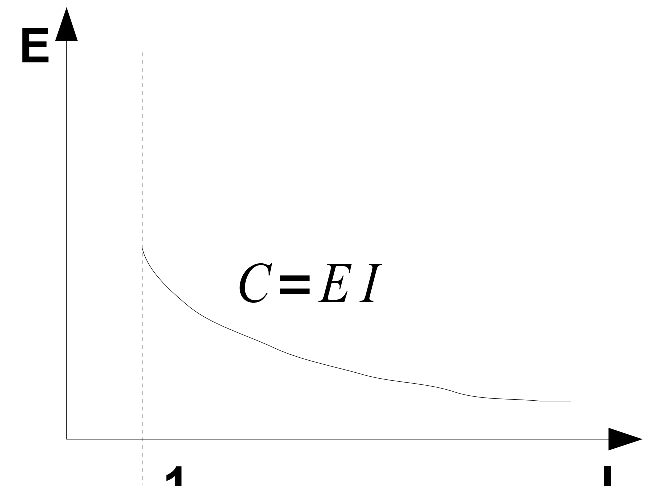
  If stabilizing solution not found, then increase q and re-do the routine

  - Deflation of Krylov Subspace

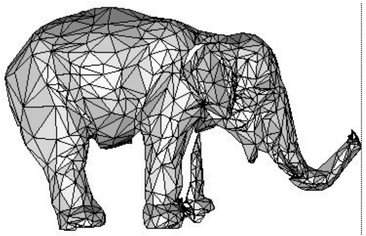
- Coefficient: **E**=computation effort; **I**=inaccuracy $I=1-\ln(a)$

  $C=E*I$ or $C=E*I^2$ (power: importance in our specific optimization problem)

  QUESTION: **What is 'C' for the problem being tackled?**



Processing Flow



extractors

$$E\dot{x} = Ax + Bu, \quad y = C^T x$$

optimization formulation

LMI solvers ⟷ Algorithm 1

(U,V) $\rightarrow$ $E_{uv}\dot{z} = A_{uv}z + B_v u, \quad y = (C^T)_v z$

EXTENTION TO PASSIVITY

- A stronger condition for stability
- Simply by adding one constraint

$$U^T B = V^T C$$

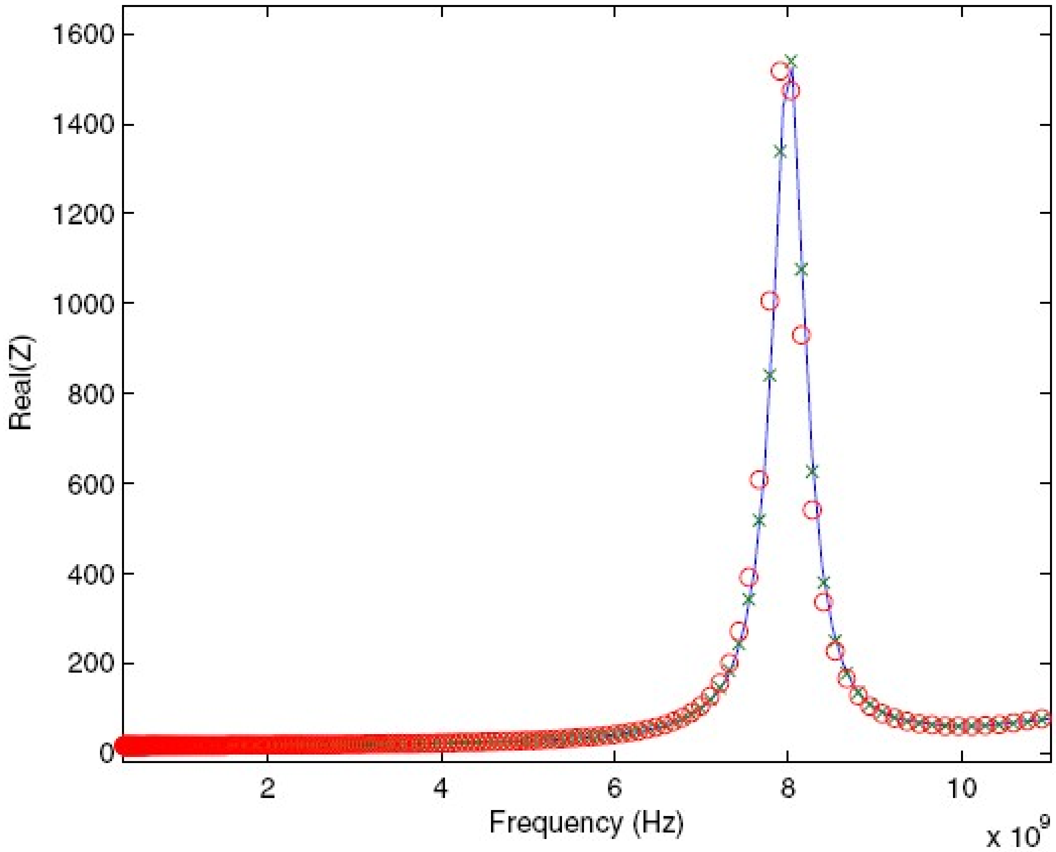
  - To guarantee no energy generated by the system
- However, still the same optimization problem

$$\min_{U_\delta, Q_1 > 0, Q_2 > 0} \|U_\delta\|$$
$$s.t. \quad U_\delta^T E_{vp} = Q_1 - \Delta Q_1$$
$$U_\delta^T A_{vp} + U_\delta^T A_{vp}^T = -Q_2 - \Delta Q_2$$
$$U^T B = V^T C$$

Experimental results

3*3 power grid

- solver(EMQS)
  - Unstable;N=1566
  - q=10
- Galerkin Projection(cross)
  - Look nice but potentially unstable



Further consideration

- Deflation of Krylov Subspace[ref.PROMIS]:

$$Kr(A,k,q)=colspan[k, Ak, A^2 k, \ldots, A^q k]$$

  - Reduced model unchanged w.r.t. the chosen of V.
- Summary of different methods(idear, pros and ons)extractors and LMI solver

Conclusion and Discussion

Stable model order reduction through a (left,right) projection pair

- Fix the right projection matrix, **optimize** the left one for best accuracy **while preserve stability and passivity**
- Also fit for *indefinite or unstable* system
- The efficiency lies in solving a LMI independent of the size of original large system

reference

- [prima]A.Odabasioglu,M.Celik. PRIMA:passive reduced order interconnect macromodeling algorithm. IEEE Trans
- [ref.11]M.Vidyagasar. Nonlinear Systems Analysis. Prentice Hall
- [ref.12]S.Sastry. Nonlinear Systems: Analysis, Stability, and Control. Springer
- [ref.6]Freund. Krylov-subspace methods for reduced-order modeling in circuit simulation. J.Comput.Appl.Math