Tarski Algorithm

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Motivation

Automatic:

1. Generation of true assertions.
   "...the most ignorant Person at a reasonable Charge, and with a little bodily Labour, may write Books in Philosophy, Poetry, Politicks, Law, Mathematics and Theology, without the least Assistance from Genius or Study."

   Jonathan Swift — Gulliver's Travels
2. Proof of assertions.
   - Verification of systems of polynomial inequalities and equations.
   - Proof of finite geometry problems.

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Alfred Tarski

January 14, 1902, Warsaw, Poland – October 26, 1983, Berkeley, California

 

Alfred Tarski

- When Tarski entered the University of Warsaw in 1918, he intended to study biology
- Tarski's first paper, published when he was only 19 years old, was on set theory
- He left Poland in August 1939, on the last ship to sail from Poland for the United States
- Tarski supervised 24 Ph. Ds and coauthored over 100 books and papers.

Part I

Geometry

Getting interested

Geometry

Problems in geometry:

- calculation
- construction
- proof
- . . .

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Language of geometry

Objects:

- points
- lines
- circles
- . . .

Language of geometry

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Language of geometry

Relations:

- "Point A is on the line l"
  , $\text{OnLine}(A, l)$
- "Point A is on the circle O"
  , $\text{OnCircle}(A, O)$
- "The distance between A and B equals distance between C and D"
  , $\text{EqDistance}(A, B, C, D)$
- ...

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- $\dots$

Language of geometry

Axioms:

- "For any points A, B there are exists line l, such as A and B are on l"

$$\forall A \forall B \exists l \{\mathrm{OnLine}(A, l) \& \mathrm{OnLine}(B, l)\}$$

- "If points A and B both lies on lines l and m, and if A and B are different, then l and m coincides."

$$\forall A \forall B \forall l \forall m \{A \neq B \& \mathrm{OnLine}(A, l) \& \mathrm{OnLine}(B, l) \& \\ \& \mathrm{OnLine}(A, m) \& \mathrm{OnLine}(B, m) \Rightarrow l = m\}$$

- ...

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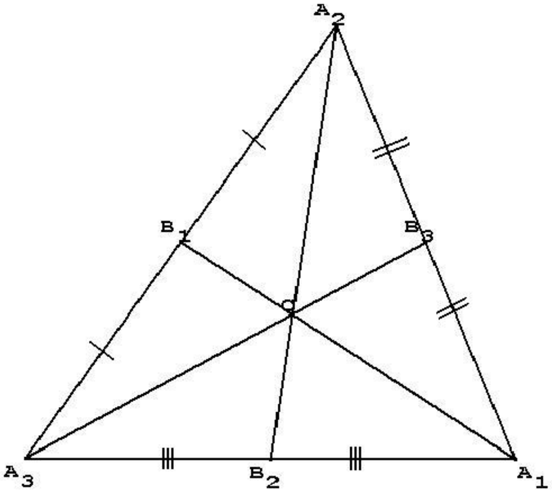
- ...

Sample

Theorem. Medians of triangle intersect at one point.

Graphical version:



Sample

Theorem. For any three mutually different points A_1, A_2 and A_3 there are four points B_1, B_2, B_3 and C and six lines l_1, l_2, l_3, m_1, m_2 and m_3 such as:

$$\mathrm{OnLine}(A_2, l_1) \& \mathrm{OnLine}(A_3, l_1) \& \mathrm{OnLine}(B_1, l_1) \&$$
$$\mathrm{OnLine}(A_1, l_2) \& \mathrm{OnLine}(A_1, l_2) \& \mathrm{OnLine}(B_2, l_2) \&$$
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$$\mathrm{OnLine}(A_3, m_3) \& \mathrm{OnLine}(B_3, m_3) \& \mathrm{OnLine}(C, m_3) \&$$
$$\mathrm{EqDistance}(A_1, B_2, B_2, A_3) \&$$
$$\mathrm{EqDistance}(A_2, B_1, B_1, A_3) \&$$
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Theorem. For any three mutually different points A_1, A_2 and A_3 there are four points B_1, B_2, B_3 and C and six lines l_1, l_2, l_3, m_1, m_2 and m_3 such as:

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$$\mathrm{OnLine}(A_1, m_1) \& \mathrm{OnLine}(B_1, m_1) \& \mathrm{OnLine}(C, m_1) \&$$
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Language of geometry - simplified

Objects: only points

Language of geometry - simplified

Objects: only points

Language of geometry - simplified

Relations:

- "Points A, B and C are on the same line" $\mathrm{OnLine}(A, B, C)$
- "Points A and B are on the same circle with center C" $\mathrm{OnCircle}(A, B, C)$
- "The distance between A and B equals distance between C and D" $\mathrm{EqDistance}(A, B, C, D)$
- $\ldots$

Sample

Theorem. For any three points A_1, A_2 and A_3 there are four points B_1, B_2, B_3 and C such as:

$$A_1 \neq A_2 \& A_1 \neq A_3 \& A_2 \neq A_3 \Rightarrow$$
$$\mathrm{OnLine}(A_1, A_2, B_3) \& \mathrm{OnLine}(A_2, A_3, B_1) \& \mathrm{OnLine}(A_1, A_3, B_2) \&$$
$$\mathrm{OnLine}(A_1, B_1, C) \& \mathrm{OnLine}(A_2, B_2, C) \& \mathrm{OnLine}(A_3, B_3, C) \&$$
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Part II

Algebra

Prerequisites for solution

Change

Point $A \leftrightarrow$ pair of reals x and y

Sample

Theorem. For any real numbers $a_{1,x}$, $a_{1,y}$, $a_{2,x}$, $a_{2,y}$, $a_{3,x}$, $a_{3,y}$, there are such real $b_{1,x}$, $b_{1,y}$, $b_{2,x}$, $b_{2,y}$, $b_{3,x}$, $b_{3,y}$, c_x and c_y:

$$(a_{1,x} \neq a_{2,x} \vee a_{1,y} \neq a_{2,y}) \& (a_{1,x} \neq a_{3,x} \vee a_{1,y} \neq a_{3,y}) \&$$
$$(a_{2,x} \neq a_{3,x} \vee a_{2,y} \neq a_{3,y}) \Rightarrow$$
$$\mathrm{OnLine}(a_{1,x}, a_{1,y}, a_{2,x}, a_{2,y}, b_{3,x}, b_{3,y}) \&$$
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Change

$$\text{OnLine}(a_x, a_y, b_x, b_y, c_x, c_y) \longleftrightarrow a_x b_y + a_y c_x + b_x c_y - a_x c_y - a_y b_x - b_y c_x = 0$$

$$\text{EqDistance}(a_x, a_y, b_x, b_y, c_x, c_y, d_x, d_y) \longleftrightarrow (a_x - b_x)^2 + (a_y - b_y)^2 = (c_x - d_x)^2 + (c_y - d_y)^2$$

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$$(a_{2,x} \neq a_{3,x} \vee a_{2,y} \neq a_{3,y}) \Rightarrow$$

$$a_{1,x}a_{2,y} + a_{1,y}b_{3,x} + a_{2,x}b_{3,y} - a_{1,x}b_{3,y} - a_{1,y}a_{2,x} - a_{2,y}b_{3,x} = 0 \ \&$$

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$$a_{1,x}b_{1,y} + a_{1,y}c_x + b_{1,x}c_y - a_{1,x}c_y - a_{1,y}b_{1,x} - b_{1,y}c_x = 0 \ \&$$

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$$(a_{1,x} - b_{2,x})^2 + (a_{1,y} - b_{2,y})^2 = (b_{2,x} - a_{3,x})^2 + (b_{2,y} - a_{3,y})^2 \ \&$$

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Language A

Language A has:

- *notation* for all rational numbers
- *variables* for real numbers
- *operations* of addition and multiplication for constructing polynomials
- *unary predicates* $= 0$, > 0, < 0, so the elementary formulas have forms $\mathrm{P} = 0$, $\mathrm{P} > 0$, and $\mathrm{P} < 0$
- *logical connectives* $\&$, $\vee$, $\neg$, $\Rightarrow$
- *quantifiers* $\forall$ and $\exists$

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Example

Assertion

$$x^2y + 4xy^3 > (x - y)^2 \;\&\; xy = 3x + 2y$$

is absurd

Example

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$$x^2y + 4xy^3 > (x - y)^2 \ \& \ xy = 3x + 2y$$

is absurd

Example

$$x^2y + 4xy^3 > (x-y)^2 \ \& \ xy = 3x + 2y$$

- for $x = 4$ and $y = 5$?
- for any x and y?
- do such x and y exist?

Example

$$x^2y + 4xy^3 > (x - y)^2 \ \& \ xy = 3x + 2y$$

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Example

$x^2y + 4xy^3 > (x-y)^2 \ \&\ xy = 3x + 2y$

- for $x = 4$ and $y = 5$?
- for any x and y?
- do such x and y exist?

Formulas

Open:

$$(x \equiv y) \to (z \& (y \vee \neg x))$$

Partially open:

$$\forall x(\exists y(x \vee y) \& (\forall z((x \& y) \to z)))$$

Closed:

$$\forall x \exists y(\forall z(x \vee y \vee z) \& \exists z(x \& \neg y \& \neg z))$$

Formulas

Open:

$$(x \equiv y) \to (z \& (y \vee \neg x))$$

Partially open:

$$\forall x (\exists y (x \vee y) \& (\forall z ((x \& y) \to z)))$$

Closed:

$$\forall x \exists y (\forall z (x \vee y \vee z) \& \exists z (x \& \neg y \& \neg z))$$

Formulas

Open:

$$(x \equiv y) \to (z \& (y \vee \neg x))$$

Partially open:

$$\forall x (\exists y (x \vee y) \& (\forall z ((x \& y) \to z)))$$

Closed:

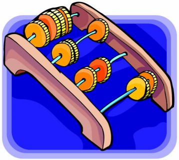
$$\forall x \exists y (\forall z (x \vee y \vee z) \& \exists z (x \& \neg y \& \neg z))$$

Theorem of A.Tarski

Theorem. (Alfred Tarski) There exists an algorithm for deciding for a given arbitrary closed formula of the language A whether the formula is true or not.

Part III

Computer Science



Solution

Plan

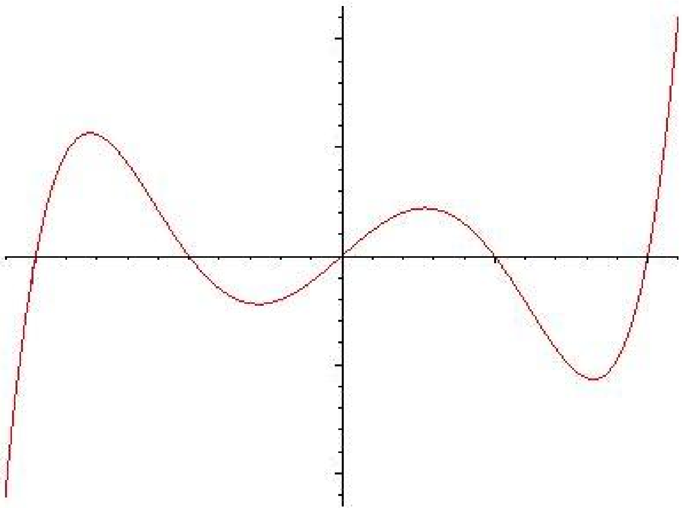
Construction:

1. Particular case of one variable polynomials
2. General case, handled with induction

Part 1

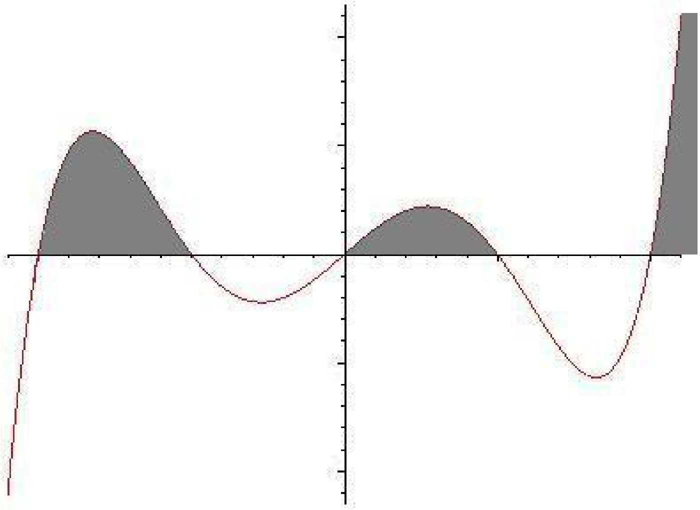
Particular case of one variable polynomials

Observation

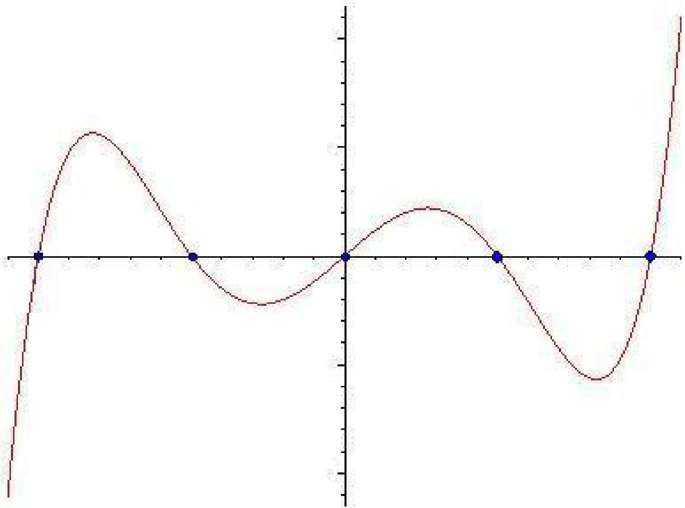


Observation

$f(x) > 0$



Observation



"Algorithm" of Tarski for formula $Qx\Phi(x)$

1. Produce the list $\mathrm{P}_1(x), \ldots, \mathrm{P}_\mathrm{k}(x)$ of all polynomials which occur in $\Phi(x)$
2. Compute the set $\mathfrak{N} = \{x_0, \ldots, x_n\}$ consiting of all real roots of all polynomials $\mathrm{P}_1(x), \ldots, \mathrm{P}_\mathrm{k}(x)$ which are different from identical zero; assume $x_0 < x_1 < \cdots < x_{n-1} < x_n$
3. Extend the set $\mathfrak{N}$ to the set $\mathfrak{M} = \{y_0, \ldots, y_m\} \supset \mathfrak{N}$ such that
   - for every i, such that $0 < i \leq n$, there exists j, such that $0 < i \leq n$ and $x_{i-1} < y_j < x_i$
   - for every i, such that $0 \leq i \leq m$, $y_0 < x_i$
   - for every i, such that $0 \leq i \leq n$, $x_i < y_m$
4. Formula $\exists x\Phi(x)$ is true if and only if $\Phi(y_0) \bigvee \cdots \bigvee \Phi(y_m)$
   Formula $\forall x\Phi(x)$ is true if and only if $\Phi(y_0)\& \ldots \&\Phi(y_m)$.

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   Formula $\forall x\Phi(x)$ is true if and only if $\Phi(y_0)\& \ldots \&\Phi(y_m)$.

Tarski Algorithm : 1st version

"Algorithm" of Tarski for formula $Qx\Phi(x)$

1. Produce the list $\mathrm{P}_1(x), \ldots, \mathrm{P}_\mathrm{k}(x)$ of all polynomials which occur in $\Phi(x)$
2. Compute the set $\mathfrak{N} = \{x_0, \ldots, x_n\}$ consiting of all real roots of all polynomials $\mathrm{P}_1(x), \ldots, \mathrm{P}_\mathrm{k}(x)$ which are different from identical zero; assume $x_0 < x_1 < \cdots < x_{n-1} < x_n$
3. Extend the set $\mathfrak{N}$ to the set $\mathfrak{M} = \{y_0, \ldots, y_m\} \supset \mathfrak{N}$ such that
   - for every i, such that $0 < i \leq n$, there exists j, such that $0 < i \leq n$ and $x_{i-1} < y_j < x_i$
   - for every i, such that $0 \leq i \leq m$, $y_0 < x_i$
   - for every i, such that $0 \leq i \leq n$, $x_i < y_m$
4. Formula $\exists x\Phi(x)$ is true if and only if $\Phi(y_0) \bigvee \cdots \bigvee \Phi(y_m)$
   Formula $\forall x\Phi(x)$ is true if and only if $\Phi(y_0)\& \ldots \&\Phi(y_m)$.

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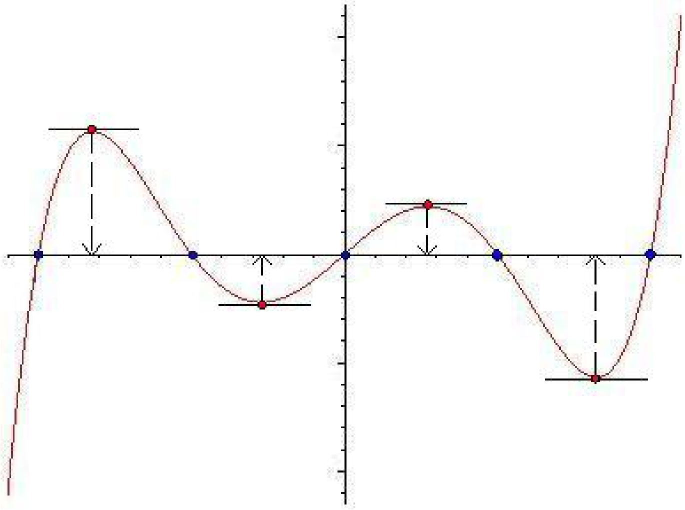
Observation



"Algorithm" of Tarski for formula $Qx\Phi(x)$

1. Produce the list $\mathrm{P_1}(x), \dots, \mathrm{P_k}(x)$ of all polynomials which occur in $\Phi(x)$ ($\mathrm{P_i}(x) \not\equiv 0$)
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4. Extend the set $\mathfrak{N}$ to the set $\mathfrak{M} = \{x_{-\infty}, x_0, x_1, \dots, x_n, x_{+\infty}\}$ where $x_{-\infty}$ and $x_{+\infty}$ are such numbers that $x_{-\infty} < x_0 < x_1 < \dots < x_{n-1} < x_n < x_{+\infty}$
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Tarski Algorithm : 2nd version

"Algorithm" of Tarski for formula $Qx\Phi(x)$

1. Produce the list $\mathrm{P_1}(x), \ldots, \mathrm{P_k}(x)$ of all polynomials which occur in $\Phi(x)$ ($\mathrm{P_i}(x) \not\equiv 0$)
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Tarski Table

	$x_{-\infty}$	x_0	$\dots$	x_j	$\dots$	x_n	$x_{+\infty}$
$\mathrm{P}_0(x)$	$\mathrm{P}_0(x_{-\infty})$	$\mathrm{P}_0(x_0)$	$\dots$	$\mathrm{P}_0(x_j)$	$\dots$	$P_0(x_n)$	$P_0(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{i}(x)$	$\mathrm{P}_\mathrm{i}(x_{-\infty})$	$\mathrm{P}_\mathrm{i}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_n)$	$\mathrm{P}_\mathrm{i}(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{k}(x)$	$\mathrm{P}_\mathrm{k}(x_{-\infty})$	$\mathrm{P}_\mathrm{k}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_n)$	$\mathrm{P}_\mathrm{k}(x_{+\infty})$

$$x_{-\infty} < x_0 < \dots < x_j \dots < x_n < x_{+\infty}$$

$$\forall j \exists i \{\mathrm{P}_\mathrm{i}(x) \not\equiv 0 \& \mathrm{P}_\mathrm{i}(x_j) = 0\}$$

$$\forall i \forall x \{(\mathrm{P}_\mathrm{i}(\mathrm{x}) \not\equiv 0 \& \mathrm{P}_\mathrm{i}(\mathrm{x}) = 0) \Rightarrow \exists j \{x = x_j\}\}$$

Tarski Table

	$x_{-\infty}$	x_0	$\dots$	x_j	$\dots$	x_n	$x_{+\infty}$
$\mathrm{P}_0(x)$	$\mathrm{P}_0(x_{-\infty})$	$\mathrm{P}_0(x_0)$	$\dots$	$\mathrm{P}_0(x_j)$	$\dots$	$P_0(x_n)$	$P_0(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{i}(x)$	$\mathrm{P}_\mathrm{i}(x_{-\infty})$	$\mathrm{P}_\mathrm{i}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_n)$	$\mathrm{P}_\mathrm{i}(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{k}(x)$	$\mathrm{P}_\mathrm{k}(x_{-\infty})$	$\mathrm{P}_\mathrm{k}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_n)$	$\mathrm{P}_\mathrm{k}(x_{+\infty})$

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$\mathrm{P}_0(x)$	$\mathrm{P}_0(x_{-\infty})$	$\mathrm{P}_0(x_0)$	$\dots$	$\mathrm{P}_0(x_j)$	$\dots$	$P_0(x_n)$	$P_0(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{i}(x)$	$\mathrm{P}_\mathrm{i}(x_{-\infty})$	$\mathrm{P}_\mathrm{i}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{i}(x_n)$	$\mathrm{P}_\mathrm{i}(x_{+\infty})$
$\vdots$							
$\mathrm{P}_\mathrm{k}(x)$	$\mathrm{P}_\mathrm{k}(x_{-\infty})$	$\mathrm{P}_\mathrm{k}(x_0)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_j)$	$\dots$	$\mathrm{P}_\mathrm{k}(x_n)$	$\mathrm{P}_\mathrm{k}(x_{+\infty})$

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Tarski Algorithm : 3rd version

"Algorithm" of Tarski for formula $Qx\Phi(x)$

1. Produce the list $\mathrm{P_1(x), \ldots, P_k(x)}$ of all polynomials which occur in $\Phi(x)$ $(\mathrm{P_i}(x) \not\equiv 0)$
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Tarski Table

Semisimplified Tarski table for polynomials $\mathrm{P}_0(x), \ldots, \mathrm{P}_\mathrm{k}(x)$

	$-\infty$	x_0	$\ldots$	x_j	$\ldots$	x_n	$+\infty$
$\mathrm{P}_0(x)$							
$\vdots$							
$\mathrm{P}_\mathrm{i}(x)$				t_{ij}			
$\vdots$							
$\mathrm{P}_\mathrm{k}(x)$							

$$t_{ij} = \begin{cases} -, & \mathrm{P}_\mathrm{i}(x_j) < 0 \\ 0, & \mathrm{P}_\mathrm{i}(x_j) = 0 \\ +, & \mathrm{P}_\mathrm{i}(x_j) > 0 \end{cases}$$

Tarski Table

Simplified Tarski table for polynomials $P_0(x), \ldots, P_k(x)$

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_k(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$

System saturation

Definition. A system of functions is called *semisaturated*, if with each function the system contains its derivative.

Lemma. Every finite system of polynomials can be extended to a finite semisaturated system of polynomials.

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$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_k(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$

Lemma. If the system of polynomials $P_0(x), \ldots, P_k(x)$ is semisaturated and $P_i(x) \not\equiv 0$, then the ith row cannot contain 0 in two consequetive cells.

System saturation

Definition. A semisaturated system of polynomials $\mathrm{P_0}(x), \ldots, \mathrm{P_n}(x)$ is called *saturated* if for each its two polynomials $\mathrm{P_k}(x)$ and $\mathrm{P_l}(x)$ such that

$$0 < \mathrm{degree}(P_l(x)) \leq \mathrm{degree}(\mathrm{P_k}(x)),$$

the system also contains the remainder $\mathrm{R}(x)$ from dividing $\mathrm{P_k}(x)$ by $P_l(x)$, i.e.,

$$\mathrm{P_k}(x) = \mathrm{Q}(x)\mathrm{P_l}(x) + \mathrm{R}(x), \mathrm{degree}(\mathrm{R}(x)) < \mathrm{degree}(\mathrm{P_l}(x))$$

System saturation

Lemma. Every finite system of polynomials can be extended to a finite saturated system of polynomials.

Lemma. If $\mathrm{P}_0(x), \ldots, P_{k-1}(x), \mathrm{P_k}(x)$ is a saturated system of polynomials and

$$\mathrm{degree}(\mathrm{P}_0(x)) \leq, \ldots, \mathrm{degree}(P_{k-1}(x)) \leq \mathrm{degree}(\mathrm{P_k}(x)),$$

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Tarski table construction

Constructing simplified Tarski system for saturated system $\mathrm{P}_0(x), \ldots, \mathrm{P}_\mathrm{k}(x)$

Tarski table construction

Basic case: all polynomials are constants

$P_0(x)$	$\pm$	$\pm$
$P_1(x)$	$\pm$	$\pm$
$\vdots$	$\vdots$	$\vdots$
$P_k(x)$	$\pm$	$\pm$

Induction step

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$							

Tarski table construction

$P_0(x)$	$\pm$ 0	$\pm$ 0	...	$\pm$ 0	...	$\pm$ 0	$\pm$ 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm$ 0	$\pm$ 0	...	$\pm$ 0	...	$\pm$ 0	$\pm$ 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	$\pm$ 0	$\pm$ 0	...	$\pm$ 0	...	$\pm$ 0	$\pm$ 0
$P_k(x)$	$\pm$						$\pm$

$$P_k(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + P_0$$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$						$\pm$

$$P_k(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + P_0$$

Tarski table construction

				x_j			
$\mathrm{P}_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_l(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{P}_{\mathrm{k}-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\mathrm{P}_{\mathrm{k}}(x)$	$\pm$			?			$\pm$

$$\mathrm{P}_{\mathrm{k}}(x_j) \lesseqqgtr 0$$

Tarski table construction

				x_j			
$\mathrm{P}_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_m(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_l(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{P}_{\mathrm{k}-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\mathrm{P}_{\mathrm{k}}(x)$	$\pm$						$\pm$

$$\mathrm{P}_{\mathrm{k}}(x) = Q(x)P_l(x) + P_m(x)$$

Tarski table construction

	x_j						
$\mathrm{P}_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_m(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_l(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{P}_{\mathrm{k}-1}(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\mathrm{P}_{\mathrm{k}}(x)$	$\pm$						$\pm$

$$\mathrm{P}_{\mathrm{k}}(x_j) = Q(x_j)P_l(x_j) + P_m(x_j) = P_m(x_j)$$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\ldots$	± 0	$\ldots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\ldots$	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\ldots$	$-$	$+$	$\ldots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm$ 0	$\pm$ 0	$\dots$	± 0		± 0	$\dots$	$\pm$ 0	$\pm$ 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm$ 0	$\pm$ 0	$\dots$	± 0		± 0	$\dots$	$\pm$ 0	$\pm$ 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm$ 0	$\dots$	± 0		± 0	$\dots$	$\pm$ 0	$\pm$ 0
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	± 0	± 0	$\dots$	± 0		± 0	$\dots$	± 0	± 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	± 0	± 0	$\dots$	$+$		± 0	$\dots$	± 0	± 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	± 0	$\dots$	± 0		± 0	$\dots$	± 0	± 0
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	± 0	± 0	$\dots$	± 0		± 0	$\dots$	± 0	± 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	± 0	± 0	$\dots$	$+$	$+$	± 0	$\dots$	± 0	± 0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	± 0	$\dots$	± 0		± 0	$\dots$	± 0	± 0
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$-$		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	$-$	$-$	± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0		± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	0		$+$	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\ldots$	± 0		± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\ldots$	$-$	0	$+$	$\ldots$	± 0	$\pm$

Tarski table construction

$\mathrm{P_0}(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{P_i}(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	0	$+$	$+$	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{P_{k-1}}(x)$	± 0	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\mathrm{P_k}(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

Tarski table construction

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\dots$	0		0	$\dots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\dots$	± 0		± 0	$\dots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\dots$	$-$	0	$+$	$\dots$	± 0	$\pm$

This is possible only if $P_i(x) \equiv 0$.

Tarski table construction

Simplified Tarski table for saturated system $P_0(x), \ldots, P_k(x)$ has been constructed!

$P_0(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_i(x)$	$\pm\ 0$	$\pm\ 0$	$\ldots$	± 0	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$P_{k-1}(x)$	± 0	$\pm\ 0$	$\ldots$	± 0	± 0	± 0	$\ldots$	$\pm\ 0$	$\pm\ 0$
$P_k(x)$	$\pm$	± 0	$\ldots$	$-$	0	$+$	$\ldots$	± 0	$\pm$

Algorithm of Tarski for formula $Qx\Phi(x)$

1. Produce list $\mathrm{Q_1}(x), \ldots, \mathrm{Q_l}(x)$ of all polynomials which occur in $\Phi(x)$ ($\mathrm{Q_i}(x) \not\equiv 0$)
2. Append the polynomial $\mathrm{Q_0}(x) = (\mathrm{Q_1}(x) \ldots \mathrm{Q_l}(x))'$
3. Extend the list to saturated system of polynomials $P_0(x), \ldots, P_k(x)$ and order them so that

$$\mathrm{degree}(\mathrm{P_0}(x)) \leq \ldots \mathrm{degree}(\mathrm{P_{k-1}}(x)) \leq \mathrm{degree}(\mathrm{P_k}(x))$$

4. Construct simplified Tarski table for $\mathrm{P_0}(x), \mathrm{P_1}(x), \ldots, \mathrm{P_m}(\mathrm{x})$ for $m = 0, 1, 2, \ldots, k$
5. Calculate logical value of $\Phi(x)$ for every column in the table
6. Formula $\exists x \Phi(x)$ is true if and only if at least one of the calculated values of $\Phi(x)$ was true;
   Formula $\forall x \Phi(x)$ is true if and only if all calculated values of of $\Phi(x)$ were true

Tarski Algorithm - final version

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Part 1 completed

The case of one quantifier system is comleted.

Part 2

Basic case of many variable polynomials

Slightly brief

Examples

Inequality with parameter:

$\mathrm{P}(a, x) > 0$

Solution:

$$\begin{cases} a < -2, \ x \in (2;3) \cup \{-a\} \\ -2 \leq a \leq 3, \ x \in [4; 7+a] \\ 3 < a, \ x \in \emptyset \end{cases}$$

Inequality with parameter:

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Examples

Assertion:

$\mathrm{A} := (\exists x : \mathrm{P}(a, x) > 0)$

$$\begin{cases} a < -2,\ \ x \in (2;3) \cup \{-a\} \Rightarrow \mathrm{A} = True \\ -2 \leq a \leq 3,\ \ x \in [4; 7 + a] \Rightarrow \mathrm{A} = True \\ 3 < a,\ \ x \in \emptyset \Rightarrow \mathrm{A} = False \end{cases}$$

Examples

Assertion:

$\mathrm{A} := (\exists x : P(a, x) > 0)$

$$\begin{cases} \mathrm{Q}_1(a) := (a < -2),\ x \in (2;3) \cup \{-a\} \Rightarrow \mathrm{A} = \mathrm{True} \\ \mathrm{Q}_2(a) := (-2 \le a \le 3),\ x \in [4; 7+a] \Rightarrow \mathrm{A} = \mathrm{True} \\ \mathrm{Q}_3(a) := (3 < a),\ x \in \emptyset \Rightarrow \mathrm{A} = \mathrm{False} \end{cases}$$

Examples

$$\mathrm{A} \equiv \mathrm{Q}_1(a) \vee \mathrm{Q}_2(a)$$

Examples

$$\exists x : bx + c = 0$$

$$(b \neq 0 \ \vee \ (b = 0 \ \& \ c = 0))$$

Examples

$$\exists x : bx + c = 0$$

$$(b \neq 0 \;\vee\; (b = 0 \;\&\; c = 0))$$

Examples

$$\exists x : ax^2 + bx + c = 0$$

$$(a \neq 0 \ \& \ b^2 \geq 4ac) \vee (b \neq 0 \vee (b = 0 \ \& \ c = 0))$$

Examples

$$\exists x : ax^2 + bx + c = 0$$

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Induction step

$$Qx : \mathrm{P}(a, x)$$

$$\mathrm{P}(a, x) = \sum_{i,j} P_{i,j} a^j x^i = \sum_i \left(\sum_j P_{i,j} a^j \right) x^j$$

Induction step

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Induction step

$$P(a,x) = \sum_i \frac{T_i(a)}{S_i(a)} x^i$$

Is it enough to construct Tarski table?

No:

- can't divide
- don't know the leading coefficient signs

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Induction step

$$\dots$$

$$\left(\frac{\mathrm{T_k}(a)}{\mathrm{S_k}(a)}x^k + \dots\right) \div \left(\frac{\mathrm{P_l(a)}}{\mathrm{Q_l(a)}}x^l + \dots\right)$$

$$\begin{cases} \mathrm{P_l(a)} = 0 \to l := l - 1 \;\; \text{(simplification)} \\ \mathrm{P_l(a)} \;\lessgtr\; 0 \to V := \frac{\mathrm{T_k}(a)}{\mathrm{S_k}(a)} \times \left(\frac{\mathrm{P_l(a)}}{\mathrm{Q_l(a)}}\right)^{-1} \end{cases}$$

$$\dots$$

Induction step - Algorithm

- Try to construct Tarki table
- Uncertainty found during division $\rightarrow$ take in excess all possible values
- Uncertainty found during Tarski table construction $\rightarrow$ take in excess all possible values
- Include temporary branch in result depending on resulting Tarski table

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- Include temporary branch in result depending on resulting Tarski table

General view

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-1} x_{n-1} \mathrm{Q}_{\mathrm{n}} x_n : \mathrm{P}_{\mathrm{n}}(x_1, \ldots, x_{n-1}, x_n)$$

$$\mathrm{Q}_{\mathrm{n}} x_n : \mathrm{P}_{\mathrm{n}}(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-1})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-2} x_{n-2} \mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-2})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-3} x_{n-3} \mathrm{Q}_{\mathrm{n}-2} x_{n-2} : \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$\mathrm{Q}_1 x_1 : \mathrm{P}_1(x_1)$$

General view

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-1} x_{n-1} \mathrm{Q}_\mathrm{n} x_n : \mathrm{P}_\mathrm{n}(x_1, \ldots, x_{n-1}, x_n)$$

$$\mathrm{Q}_\mathrm{n} x_n : \mathrm{P}_\mathrm{n}(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-1})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-2} x_{n-2} \mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-2})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-3} x_{n-3} \mathrm{Q}_{\mathrm{n}-2} x_{n-2} : \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$\mathrm{Q}_1 x_1 : \mathrm{P}_1(x_1)$$

General view

$$Q_1 x_1, \ldots, Q_{n-1} x_{n-1} Q_n x_n : P_n(x_1, \ldots, x_{n-1}, x_n)$$

$$Q_n x_n : P_n(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow P_{n-1}(x_1, \ldots, x_{n-1})$$

$$Q_1 x_1, \ldots, Q_{n-2} x_{n-2} Q_{n-1} x_{n-1} : P_{n-1}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$Q_{n-1} x_{n-1} : P_{n-1}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow P_{n-2}(x_1, \ldots, x_{n-2})$$

$$Q_1 x_1, \ldots, Q_{n-3} x_{n-3} Q_{n-2} x_{n-2} : P_{n-2}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$Q_1 x_1 : P_1(x_1)$$

General view

$$\mathrm{Q_1}x_1, \dots, \mathrm{Q_{n-1}}x_{n-1}\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \dots, x_{n-1}, x_n)$$

$$\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \dots, x_{n-1}, x_n) \leftrightarrow \mathrm{P_{n-1}}(x_1, \dots, x_{n-1})$$

$$\mathrm{Q_1}x_1, \dots, \mathrm{Q_{n-2}}x_{n-2}\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \dots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \dots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P_{n-2}}(x_1, \dots, x_{n-2})$$

$$\mathrm{Q_1}x_1, \dots, \mathrm{Q_{n-3}}x_{n-3}\mathrm{Q_{n-2}}x_{n-2} : \mathrm{P_{n-2}}(x_1, \dots, x_{n-3}, x_{n-2})$$

$$\dots$$

$$\mathrm{Q_1}x_1 : \mathrm{P_1}(x_1)$$

General view

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-1}}x_{n-1}\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \ldots, x_{n-1}, x_n)$$

$$\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow \mathrm{P_{n-1}}(x_1, \ldots, x_{n-1})$$

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-2}}x_{n-2}\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P_{n-2}}(x_1, \ldots, x_{n-2})$$

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-3}}x_{n-3}\mathrm{Q_{n-2}}x_{n-2} : \mathrm{P_{n-2}}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$\mathrm{Q_1}x_1 : \mathrm{P_1}(x_1)$$

General view

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-1}}x_{n-1}\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \ldots, x_{n-1}, x_n)$$

$$\mathrm{Q_n}x_n : \mathrm{P_n}(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow \mathrm{P_{n-1}}(x_1, \ldots, x_{n-1})$$

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-2}}x_{n-2}\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q_{n-1}}x_{n-1} : \mathrm{P_{n-1}}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P_{n-2}}(x_1, \ldots, x_{n-2})$$

$$\mathrm{Q_1}x_1, \ldots, \mathrm{Q_{n-3}}x_{n-3}\mathrm{Q_{n-2}}x_{n-2} : \mathrm{P_{n-2}}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$\mathrm{Q_1}x_1 : \mathrm{P_1}(x_1)$$

General view

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-1} x_{n-1} \mathrm{Q}_{\mathrm{n}} x_n : \mathrm{P}_{\mathrm{n}}(x_1, \ldots, x_{n-1}, x_n)$$

$$\mathrm{Q}_{\mathrm{n}} x_n : \mathrm{P}_{\mathrm{n}}(x_1, \ldots, x_{n-1}, x_n) \leftrightarrow \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-1})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-2} x_{n-2} \mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1})$$

$$\mathrm{Q}_{\mathrm{n}-1} x_{n-1} : \mathrm{P}_{\mathrm{n}-1}(x_1, \ldots, x_{n-2}, x_{n-1}) \leftrightarrow \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-2})$$

$$\mathrm{Q}_1 x_1, \ldots, \mathrm{Q}_{\mathrm{n}-3} x_{n-3} \mathrm{Q}_{\mathrm{n}-2} x_{n-2} : \mathrm{P}_{\mathrm{n}-2}(x_1, \ldots, x_{n-3}, x_{n-2})$$

$$\ldots$$

$$\mathrm{Q}_1 x_1 : \mathrm{P}_1(x_1)$$

Input:

$$\forall a_{1,x} \forall a_{1,y} \forall a_{2,x} \forall a_{2,y} \forall a_{3,x} \forall a_{3,y} \exists b_{1,x} \exists b_{1,y} \exists b_{2,x} \exists b_{2,y} \exists b_{3,x} \exists b_{3,y} \exists c_x \exists c_y :$$

$$((a_{1,x} - a_{2,x})^2 > 0 \vee ((a_{1,y} - a_{2,y})^2 > 0) \& ((a_{1,x} - a_{3,x})^2 > 0 \vee ((a_{1,y} - a_{3,y})^2 > 0) \&$$
$$((a_{2,x} - a_{3,x})^2 > 0 \vee ((a_{2,y} - a_{3,y})^2 > 0) \Rightarrow$$
$$a_{1,x}a_{2,y} + a_{1,y}b_{3,x} + a_{2,x}b_{3,y} - a_{1,x}b_{3,y} - a_{1,y}a_{2,x} - a_{2,y}b_{3,x} = 0 \ \&$$
$$a_{2,x}a_{3,y} + a_{2,y}b_{1,x} + a_{3,x}b_{1,y} - a_{2,x}b_{1,y} - a_{2,y}a_{3,x} - a_{3,y}b_{1,x} = 0 \ \&$$
$$a_{1,x}a_{3,y} + a_{1,y}b_{2,x} + a_{3,x}b_{2,y} - a_{1,x}b_{2,y} - a_{1,y}a_{3,x} - a_{3,y}b_{2,x} = 0 \ \&$$
$$a_{1,x}b_{1,y} + a_{1,y}c_x + b_{1,x}c_y - a_{1,x}c_y - a_{1,y}b_{1,x} - b_{1,y}c_x = 0 \ \&$$
$$a_{2,x}b_{2,y} + a_{2,y}c_x + b_{2,x}c_y - a_{2,x}c_y - a_{2,y}b_{2,x} - b_{2,y}c_x = 0 \ \&$$
$$a_{3,x}b_{3,y} + a_{3,y}c_x + b_{3,x}c_y - a_{3,x}c_y - a_{3,y}b_{3,x} - b_{3,y}c_x = 0 \ \&$$
$$(a_{1,x} - b_{2,x})^2 + (a_{1,y} - b_{2,y})^2 = (b_{2,x} - a_{3,x})^2 + (b_{2,y} - a_{3,y})^2 \ \&$$
$$(a_{2,x} - b_{1,x})^2 + (a_{2,y} - b_{1,y})^2 = (b_{1,x} - a_{3,x})^2 + (b_{1,y} - a_{3,y})^2 \ \&$$
$$(a_{1,x} - b_{3,x})^2 + (a_{1,y} - b_{3,y})^2 = (b_{3,x} - a_{2,x})^2 + (b_{3,y} - a_{2,y})^2$$

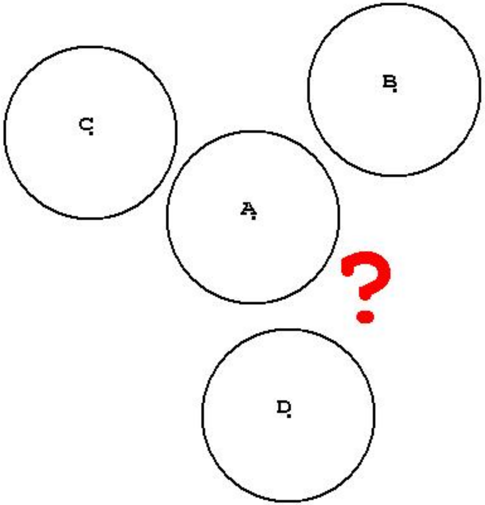
Part IV

Conclusion

Finishing

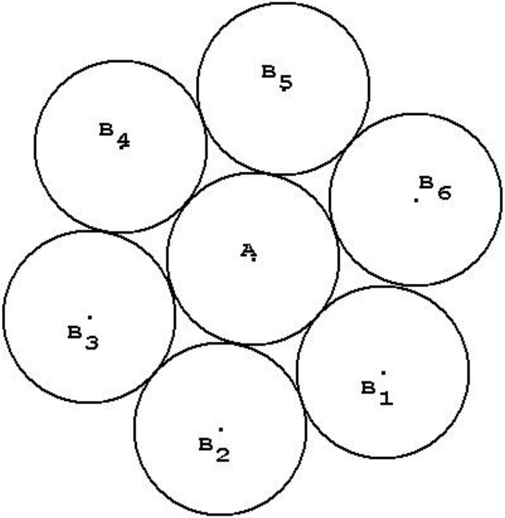
Example

2-dimension case
How many adjacent circles?



Example

2-dimension case: 6 adjacent circles



Example

3-dimension case
How many adjacent balls?

- 11
- 12
- 13

Example

3-dimension case
The answer is 12

Lower bounds

The proven lower bound — $e^{e^{x}}$

Reality

Some speedup:

1. Random points method
2. Cylindrical algebraic decomposition

Reality

Some speedup:

1. Random points method
2. Cylindrical algebraic decomposition

Reality

D. Lazard
An improved projection for cylindrical algebraic decomposition
Unpublished manuscript, 1990

Reality

$$x^2 + y^2 + z^2 < 1$$

$$\begin{cases} -1 < x < 1, \\ -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}, \\ -\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \end{cases}$$

Reality

$$x^2 + y^2 + z^2 < 1$$

$$\begin{cases} -1 < x < 1, \\ -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}, \\ -\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \end{cases}$$

Reality

Some speedup:

1. Random points method
2. Cylindrical algebraic decomposition
3. May be perfect ways still unopened?

Thank you for attention!