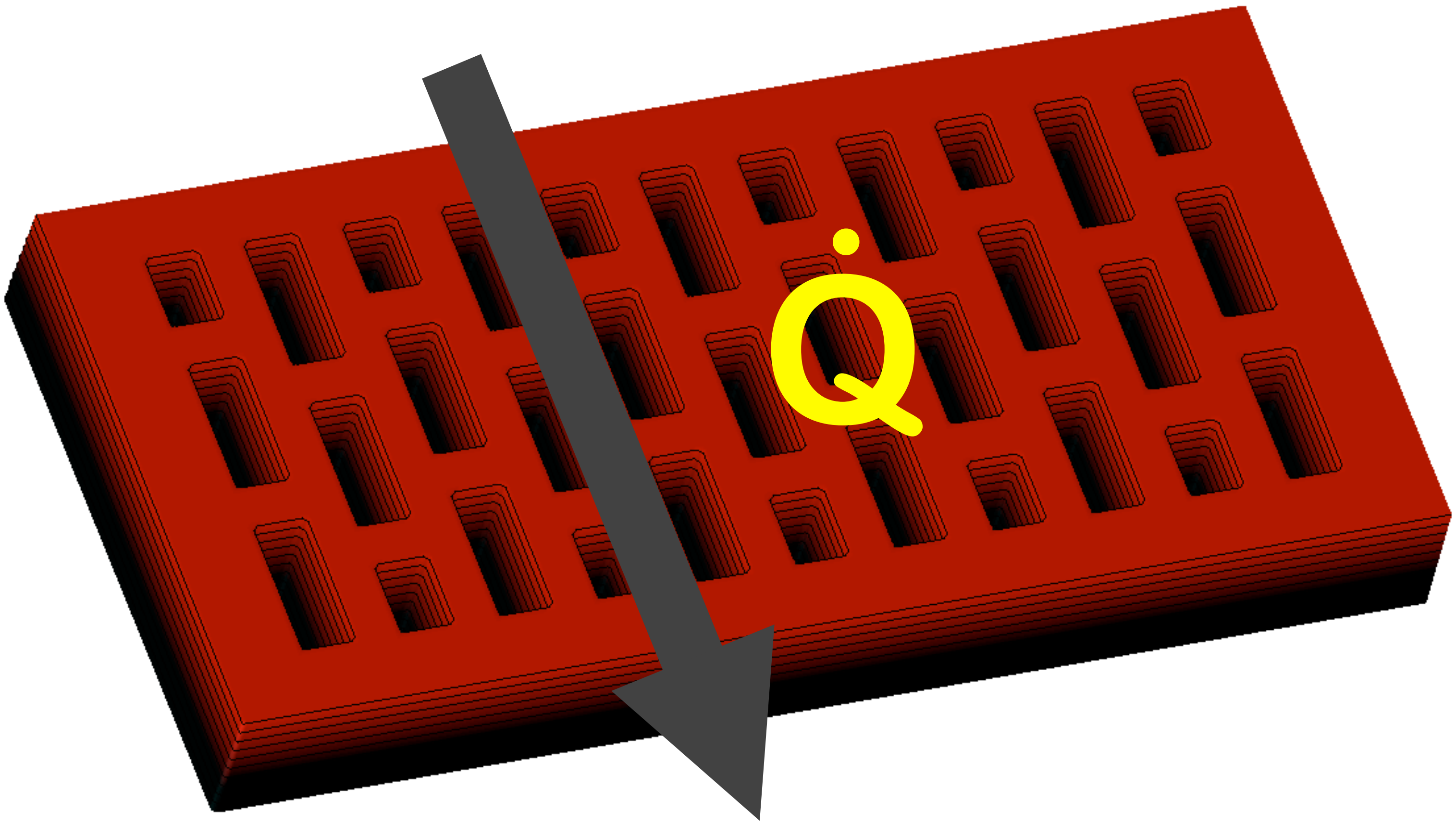
Computation of heat conduction ...

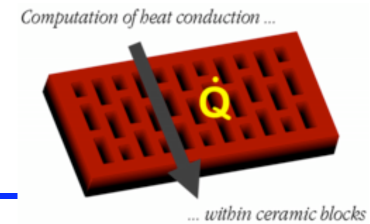


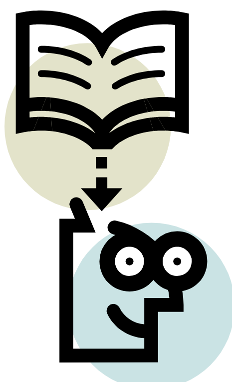
... within ceramic blocks

Overview

- **Modelling:**
  from nature to mathematics
- **Simulation:**
  solution of the mathematical problem by finite elements and multigrid method
- **Visualization of the results:**
  pictures of isothermal lines, temperature distribution and heat flow vectors

What is heat conduction?

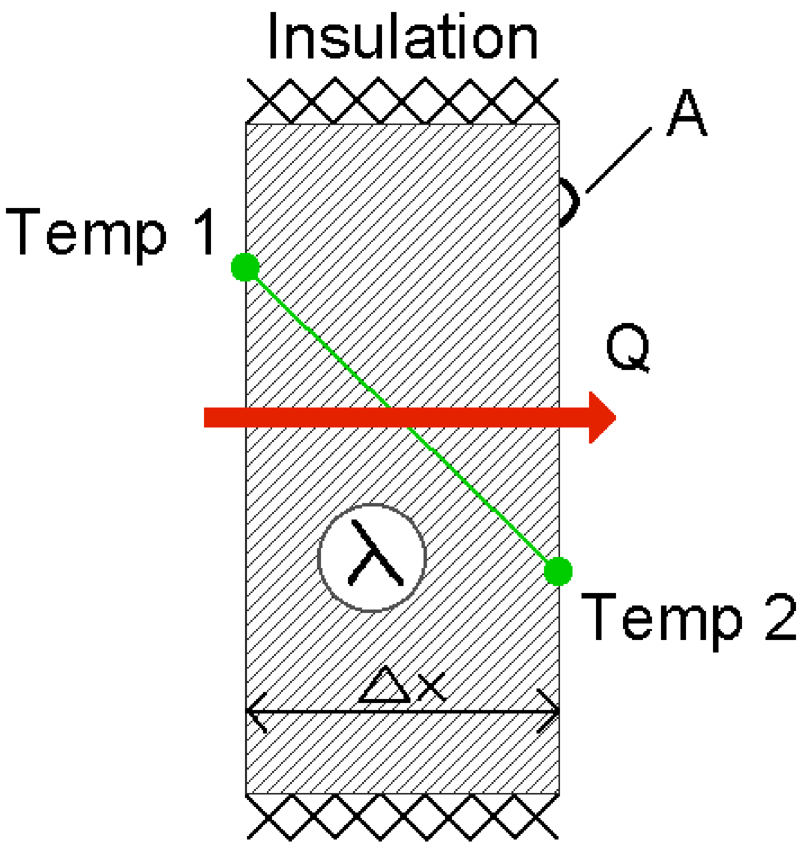




heat conduction:

diffusive transport of energy in solids, liquids and gases, caused by Brownian motion of atoms and molecules

Fourier´s law of heat conduction



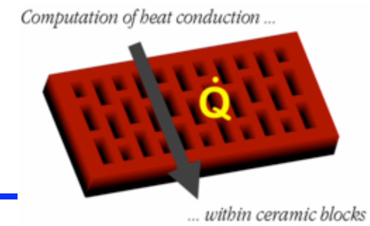
The amount Q of transferred heat is proportional to:

- temperature difference T_1-T_2
- cross-sectional area A
- period of time Δt
- inverse thickness $1 / \Delta x$

$$\Rightarrow Q = \lambda \frac{T_1 - T_2}{\Delta x} A \Delta t$$

The coefficient λ is called **thermal conductivity** and strongly depends on the material.

Fourier´s law (continued)

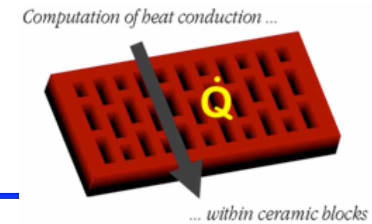


$$Q = \lambda \frac{T_1 - T_2}{\Delta x} A \Delta t$$

Because of [Q] = Joule = Watt sec, [T] = Kelvin
the unit of the thermal conductivity λ has to be W/(mK).

material	**thermal conductivity**
gold	295 W/(mK)
aluminium	230 W/(mK)
glass	1.4 W/(mK)
H_2O	0.6 W/(mK)
air	0.025 W/(mK)

Fourier´s law (continued)



transferred heat with respect to time

$$heat flow: \qquad \dot{Q} = \frac{dQ}{dt} = \lambda \frac{T_1 - T_2}{\Delta x} A \quad [W]$$

transferred heat with respect to time and area

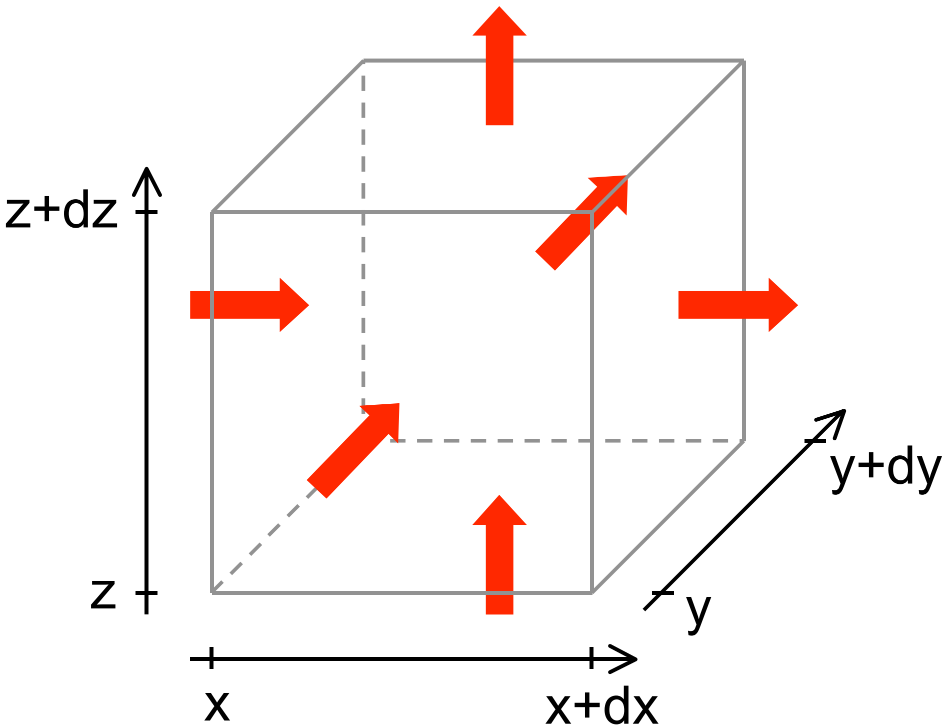
$$heat flux: \qquad \dot{q} = \frac{\dot{Q}}{A} = \lambda \frac{T_1 - T_2}{\Delta x} \quad [\frac{W}{m^2}]$$

In the limit Δx □ 0, we obtain **Fourier´s law**:

$$\dot{q} = -\lambda \nabla T$$

Derivation of Fourier´s PDE

volume element dV = dx dy dz



$$\dot{Q}(x) = \dot{q}_x \, dy \, dz,$$

$$\dot{Q}(x+dx) = (\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx) \, dy \, dz$$

Net heat entry by x-direction:

$$\dot{Q}(x) - \dot{Q}(x+dx) = -\frac{\partial \dot{q}_x}{\partial x} dx \, dy \, dz$$

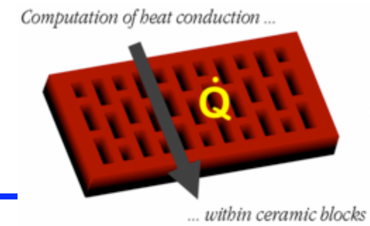
y- and z-direction accordingly:

$$-\frac{\partial \dot{q}_y}{\partial y} dx \, dy \, dz, \qquad -\frac{\partial \dot{q}_z}{\partial z} dx \, dy \, dz$$

Under steady-state conditions, the sum of all three must vanish:

$$-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} = 0$$

Fourier´s PDE (continued)



$$-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} = 0$$

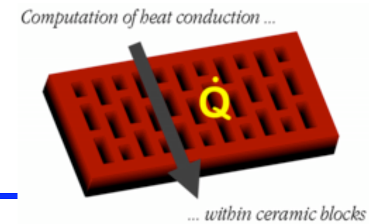
remember Fourier´s law:

$$\dot{q}_x = -\lambda \frac{\partial T}{\partial x}, \quad \dot{q}_y = -\lambda \frac{\partial T}{\partial y}, \quad \dot{q}_z = -\lambda \frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x}(\lambda \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(\lambda \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(\lambda \frac{\partial T}{\partial z}) = 0$$

or simply: $div(\lambda \nabla T) = 0$

Boundary conditions ...



... of 1st type (Dirichlet)

temperature *given:* $T = T_b$

... of 2nd type (Neumann)

heat flux *given:* $\dot{q}_b = -\lambda \frac{\partial T}{\partial n}$

... of 3rd type (mixed / Cauchy)

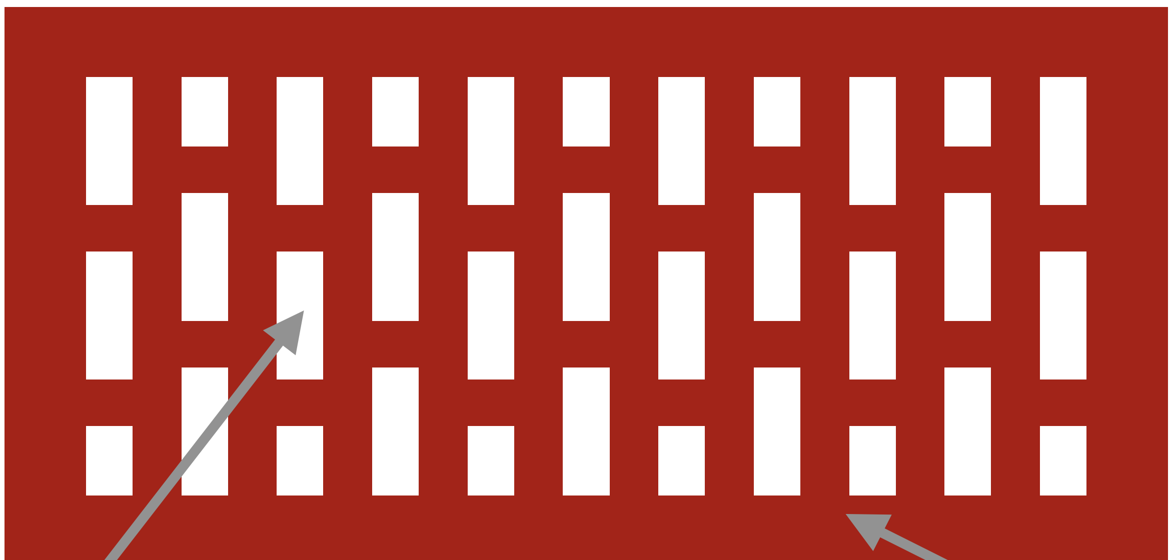
coupling of convection (+ radiation) and conduction

Mathematical model

PDE for steady-state conditions: $div(\lambda \nabla T) = 0$

$T = T_i$

$\frac{\partial T}{\partial n} = 0$



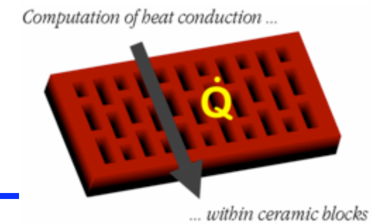
$\frac{\partial T}{\partial n} = 0$

$T = T_o$

$\Box_{air}$ = 0.025 W/(mK)

$\Box_{brick}$ = 0.61 W/(mK)

Solution in the weak sense



Let $V := \left\{ v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_D \right\}$

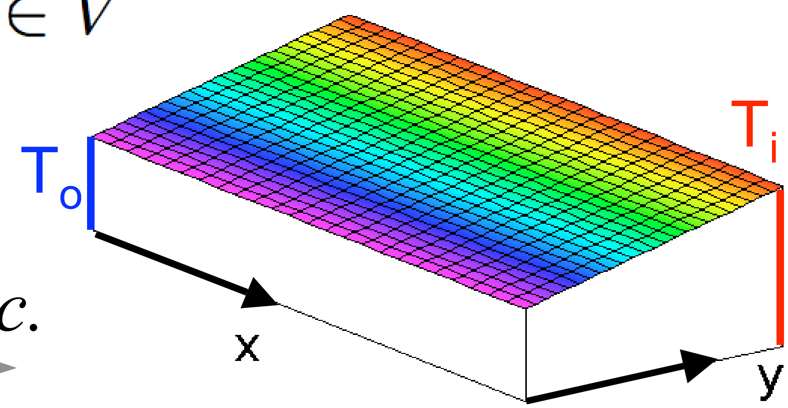
$$\Rightarrow \int_\Omega v \, div(\lambda \nabla T) d\mathbf{x} = 0 \quad \forall v \in V$$

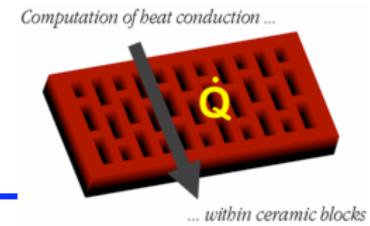
$$\Rightarrow \underbrace{\int_{\partial\Omega} v \lambda (\nabla T)^T \nu \, d\mathbf{S}}_{=0} - \int_\Omega \lambda (\nabla T)^T (\nabla v) d\mathbf{x} = 0 \quad \forall v \in V$$

Find a function T □ T_{Dir} + V so that

$$\int_\Omega \lambda (\nabla T)^T (\nabla v) d\mathbf{x} = 0 \quad \forall v \in V$$

where $T_{Dir}(x,y)$ = □ y + □
fulfills the inhom. Dirichlet b. c.





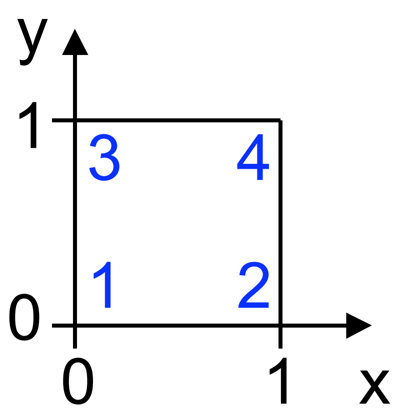
Choosing a subspace of V

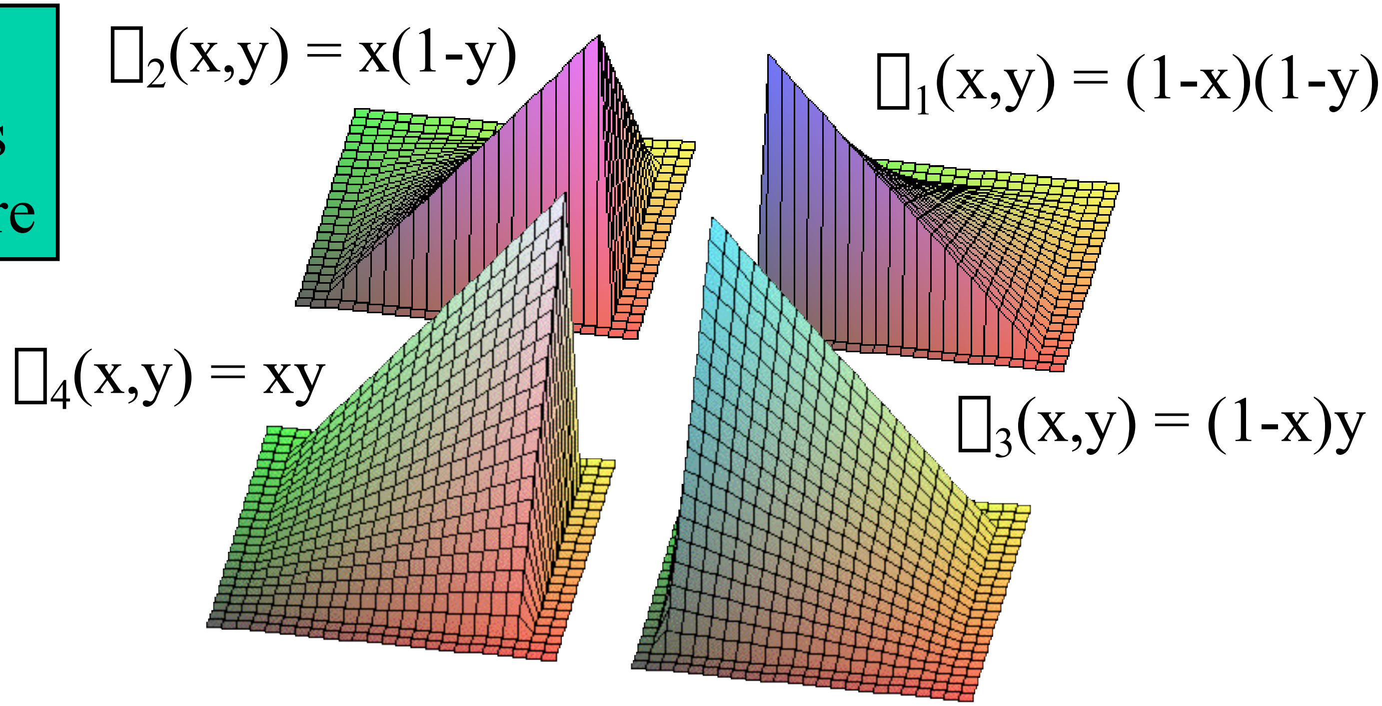
Problem: *dim* $V = \infty$

Galerkin ansatz: *Take a subspace* $S \leq V$ *with dim* $S < \infty$

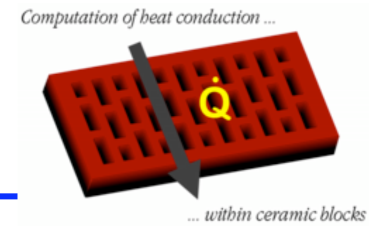
We choose: *finite element space of bilinear functions on squares*

The four local shape functions on the unit square





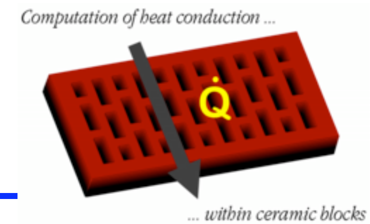
Element stiffness matrix $A^{(e)}$



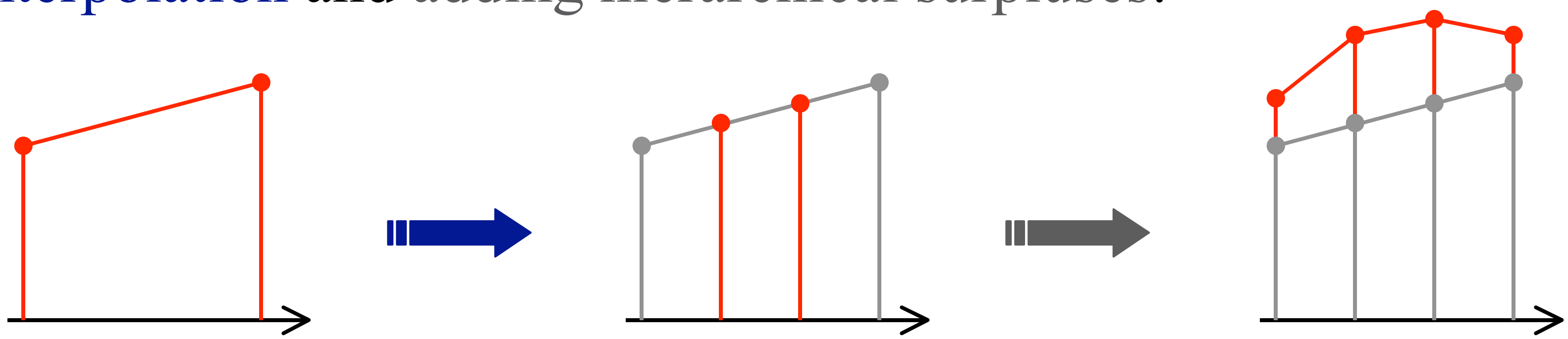
$$A_{ij}^{(e)} = \int_{(e)} \lambda^{(e)} (\nabla \varphi_j)^T (\nabla \varphi_i) d\mathbf{x}$$

$$A^{(e)} = \lambda^{(e)} / 6 \begin{pmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -2 & -1 & -1 & 4 \end{pmatrix}$$

Sketch of the algorithm



1. Go from coarsest grid level to finest by recursively performing interpolation and adding hierarchical surpluses:

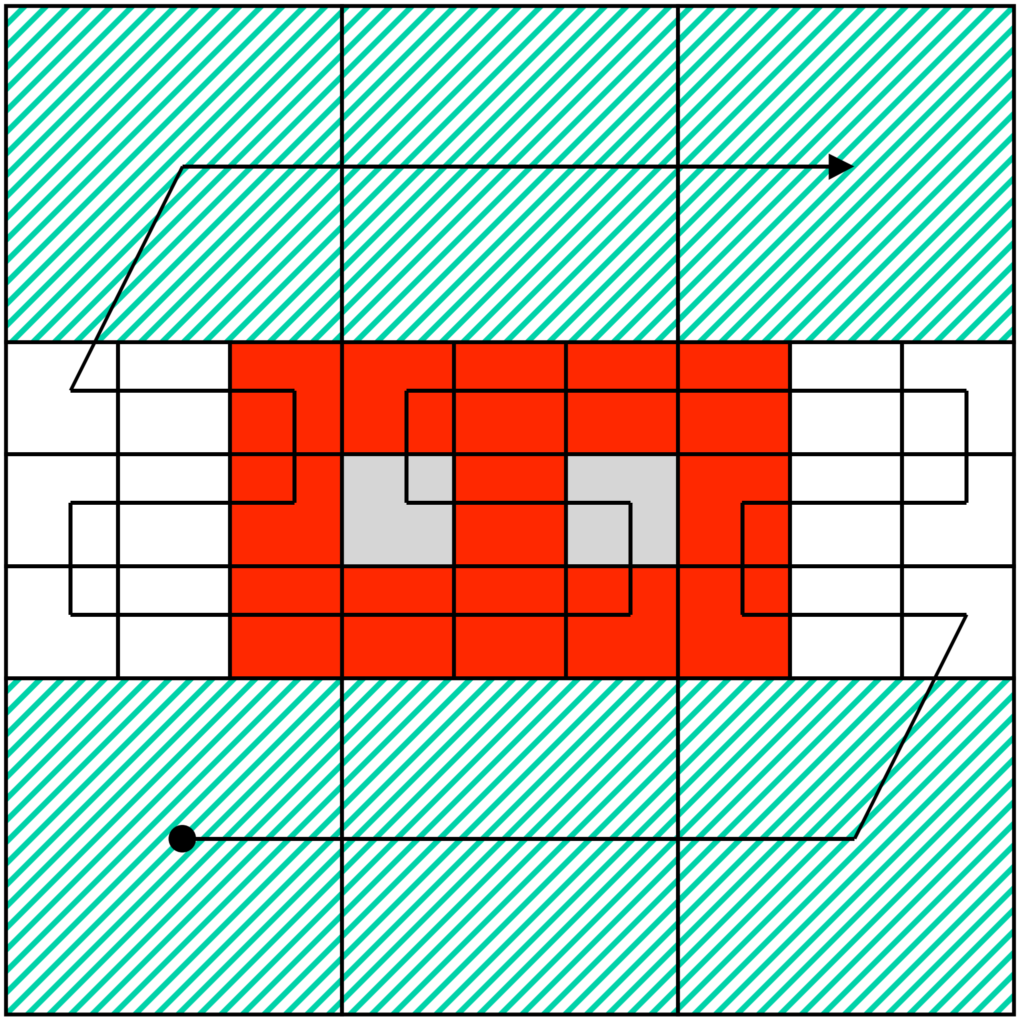


2. Compute the residual on finest grid level and perform one step of weighted Jacobi method: $T^{(k+1)} = T^{(k)} - \omega D^{-1} res^{(k)}$

approximation to the solution in k-th iteration
relaxation parameter
inverse of diagonal matrix

3. Recursively **restrict** the residual to the next coarser grid level and perform an **iteration step** there (until top level is reached).

Cell-wise processing

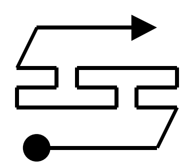


brick cell
□ = □$_{brick}$

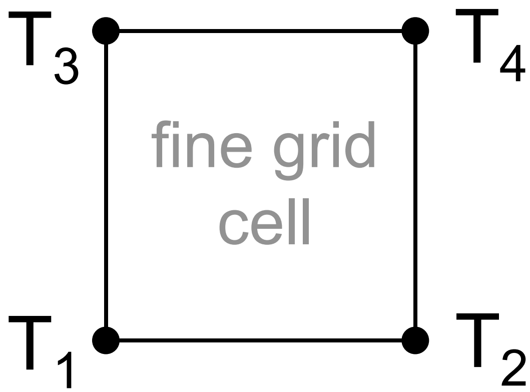
air cell
□ = □$_{air}$

Neumann cell
□ = 0.0

Dirichlet cell
(nothing to do)


Peano curve

Computation of the cell residual



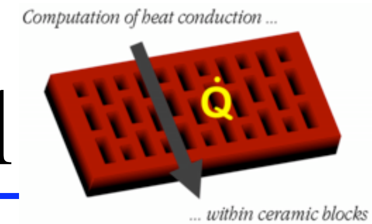
T_1, ..., T_4: current approximation

We define $T := (T_1, T_2, T_3, T_4)^T$

$$(A^{(e)}T)_i = \int_{(e)} \lambda^{(e)} (\nabla(\sum_{j=1}^{4} T_j \varphi_j))^T (\nabla \varphi_i) d\mathbf{x}$$

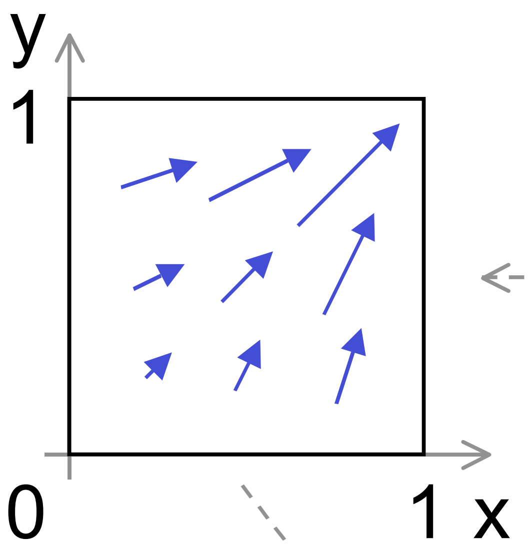
$(A^{(e)}\ T)_i$: *contribution of the cell to the residual in node i*

Interpretation of the cell residual



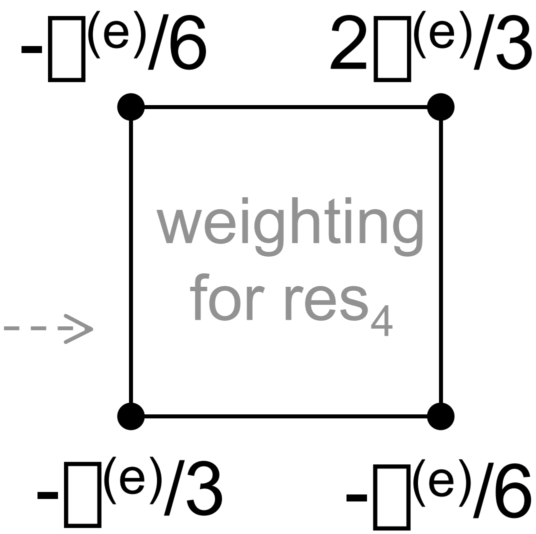
Example: $$(A^{(e)}T)_4 = \int_{(e)} \underbrace{(\lambda^{(e)} \nabla (\sum_{j=1}^{4} T_j \varphi_j))^T}_{=-\dot{q}} \begin{pmatrix} y \\ x \end{pmatrix} d\mathbf{x}$$

vector $(y,x)^T$ on the unit square

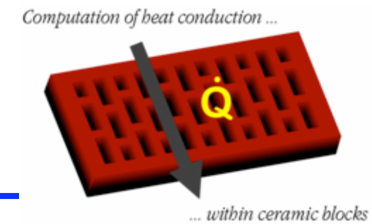


Cell residual in node i:
heat flow ***node i → cell***

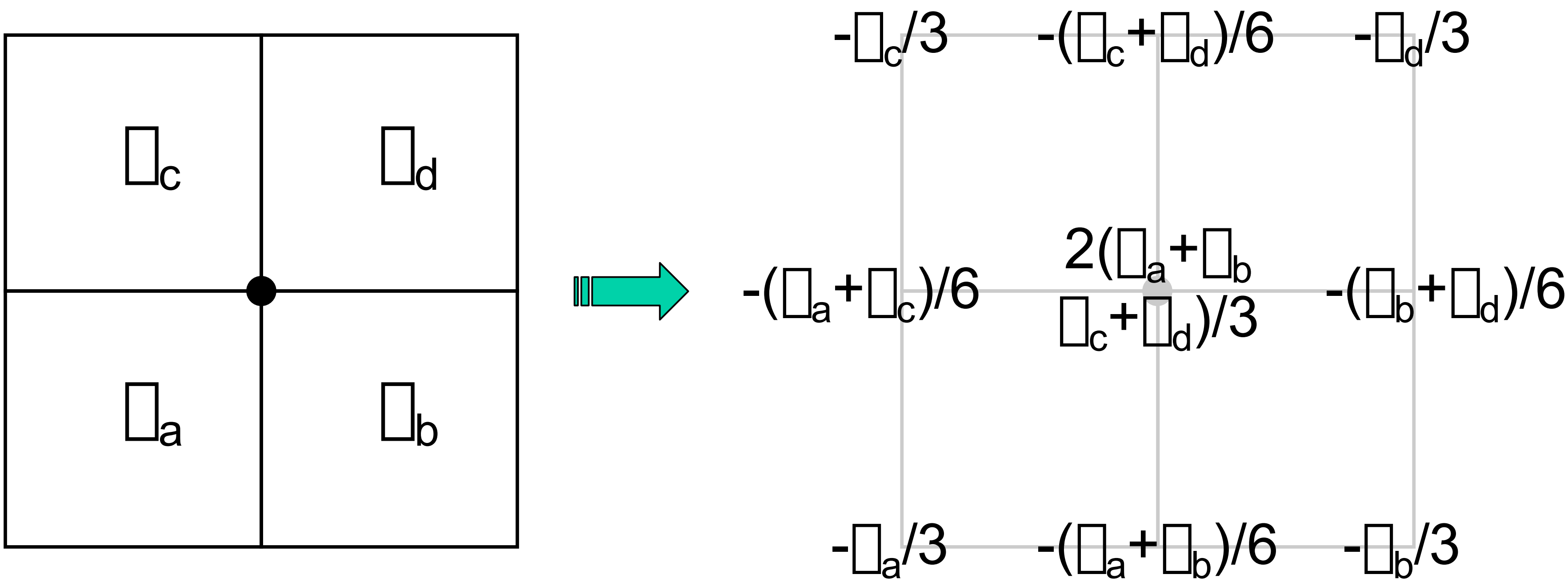
example
res in node 4



Residual assembly

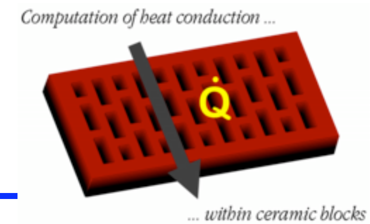


A temperature node (●) has four surrounding cells. Accordingly, the residual of one node is assembled by four cell residuals.



Assembled residual: net heat flow ***node* ➔ *surroundings***

Weighted Jacobi on finest grid



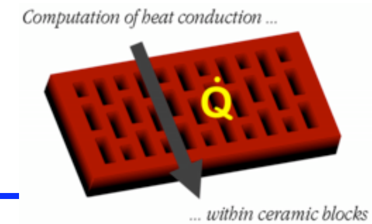
$$T_i^{(k+1)} = T_i^{(k)} - \square\, D_i^{-1}\, res_i^{(k)}$$

$$where\ D_i = 2/3\ (\square_a + \square_b + \square_c + \square_d)$$

if ($res_i^{(k)} = 0$) no correction

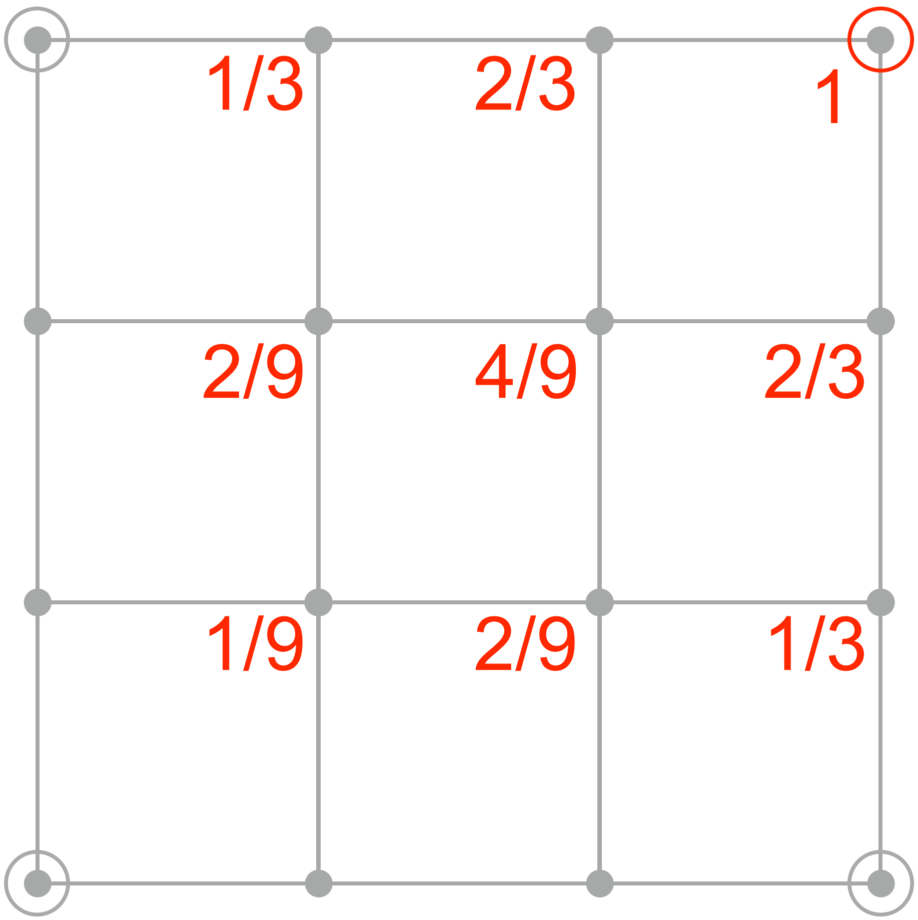
if ($res_i^{(k)} > 0$) decrease temperature

if ($res_i^{(k)} < 0$) increase temperature



Restriction of the residual

Restriction = transporting the residual to the next coarser level

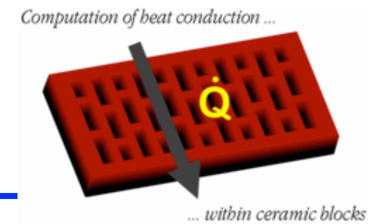


● fine grid nodes

○ coarse grid nodes

x weighting factors for calculating the coarse cell residual in the upper right corner

Correction on coarser grids



Remember the weighted Jacobi method:

$$T_i^{(k+1)} = T_i^{(k)} - \omega\, D_i^{-1}\, res_i^{(k)}$$

$res_i^{(k)}$: obtained by restriction

D_i^{-1}: In general, coarse grid cells consist of fine grid cells with different thermal conductivities.

<u>Problem:</u> how to compute D